Personalized PageRank dimensionality and algorithmic implications

Vijay G. Subramanian

ECE, University of Michigan

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Joint work with Dr. Daniel Vial, ECE, UT Austin & UIUC

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Graphs arise in many domains, and are used to understand processes, behaviors and vulnerabilities

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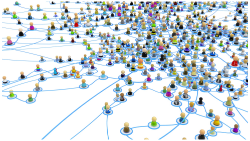
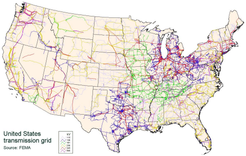


Social networks

Internet



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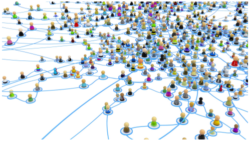

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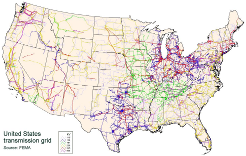

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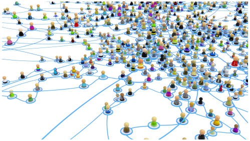

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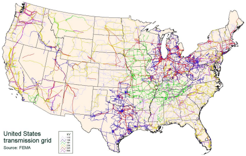

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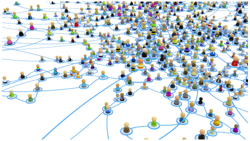

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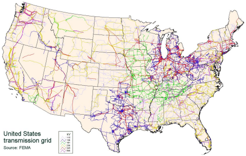

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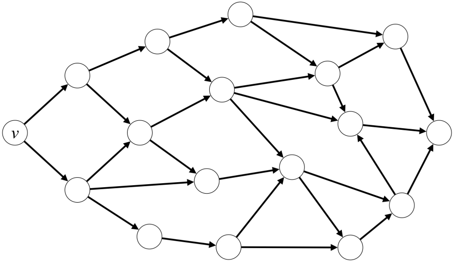
One metric used to answer these questions: Personalized PageRank (PPR)

PPR definition

Given directed graph $G = (V, E)$, let $v \in V$ and $\alpha \in (0, 1)$



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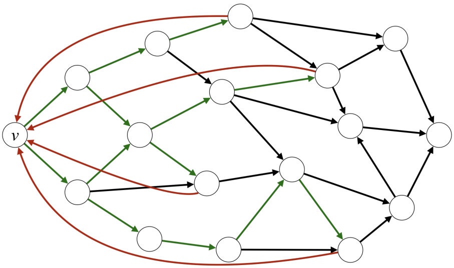
Define Markov chain $\{X_t^v\}_{t \in \mathbb{N}}$ as follows: given X_t^v,

- W.p. $(1 - \alpha)$, sample X_{t+1}^v from out-neighbors of X_t^v (random walk)
- W.p. α, set $X_{t+1}^v = v$ (jump to v)



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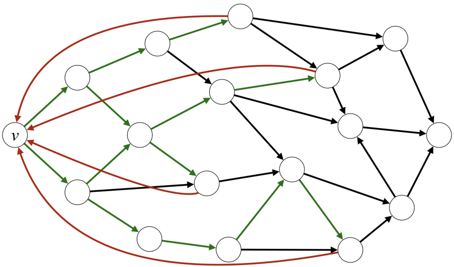
Stationary distribution $\pi_v = \{\pi_v(w)\}_{w \in V}$ called *PPR vector*



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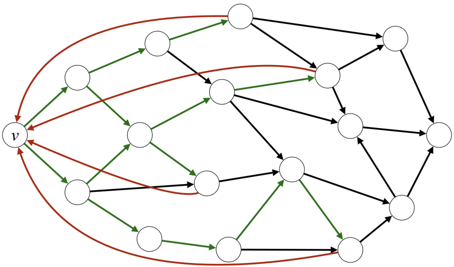
Matrix $\Pi = \{\pi_v\}_{v\in V}$ called *PPR matrix*



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(Global) PageRank and PPR

Proposed to rank websites (Page et al. 1999); many uses since

- Recommendation (Baluja et al. 2008; Gupta et al. 2013)
- Bioinformatics (Morrison et al. 2005; Freschi 2007)
- Community detection (Andersen, Chung, Lang 2006, Kloumann, Ugander, Kleinberg 2017)
- Graph similarity (Koutra, Vogelstein, Faloutsos 2013)
- ...

Definitions

Directed graph $G = (V, E)$, $V = \{1, 2, \dots, n\}$, $m = |E|$

Adjacency A, diagonal out-degree D, $P = D^{-1}A$ (row stochastic)

PPR: Perron-Frobenius eigenvector π_σ of non-negative matrix

$$P_\sigma = (1 - \alpha)P + \alpha 1_n \sigma^T \qquad (1)$$

where $\alpha \in (0, 1)$, $\sigma \in \mathbb{R}_+^n$ s.t. $\sum_{v \in V} \sigma(v) = 1$ (distribution on V)

When $\sigma = e_s$ (1 in s-th position, 0 elsewhere), denote as π_s

(Global) PageRank: $\sigma = \frac{1}{n} 1_n$

Computation of PPR

- Linear algebraic method
  - Non-negative matrix, so Perron-Frobenius theorem
  - Power method variants, $O(n^2)$ for each source
  - Directed Laplacian variants, almost linear time; Miller, Spielman, Teng, Peng.
- Probabilistic method
  - P is the transition kernel of simple random walk on G
  - Use Monte Carlo (random walks) to estimate PPR
  - $O(n \log(n))$ complexity for (Global) PageRank: Avrachenkov et al. 2007, Sarma et al. 2015
- Variational method
  - View eigenvector computation as a Bellman equation
  - Use value iteration: Andersen, Chung, Lang 2006, Andersen et al. 2008
- Hybrid schemes
  - Monte Carlo + Variational method
  - For a single entry of Π - Lofgren, Banerjee, Goel 2016

Key properties

P_σ is a Doeblin chain: Athreya, Stenflo 2003

$$\pi_s(t) = \mathbb{P}[\text{random walk from } s \text{ of length} \sim \text{geom}(\alpha) \text{ ends at } t] \quad (2)$$

$\Rightarrow$ Can sample from π_s using random walks

To estimate π_σ, suffices to estimate π_s, because

$$\pi_\sigma = \sum_{s \in V} \sigma(s) \pi_s \quad (3)$$

Renewal reward interpretation: $\pi_s(t)$ importance of t for s, as

$$\pi_s(t) \propto \mathbb{E}\left[\text{number of visits to } t \text{ on geom}(\alpha)\text{-length walk from } s\right] \quad (4)$$

PPR dimensionality question

Using Perron-Frobenius theorem, can show $\mathrm{rank}(\Pi) = |V| =: n$

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- Suggests G is $O(n)$-dimensional, but Π (derived from G) is n^2-dimensional
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Outline of talk:

1. How to quantify effective dimension of Π?
2. Can we bound this measure of dimensionality?
3. If bound "small", can we leverage it algorithmically?

Quantifying PPR dimensionality

Natural measure of effective dimension of Π:

$$\Delta(\epsilon) = \min_{\hat{\Pi}} \mathrm{rank}(\hat{\Pi}) \text{ s.t. } \|\Pi - \hat{\Pi}\| < \epsilon \tag{5}$$

Intuitively, Π low dimensional if close to low-rank matrix

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We take $\|\cdot\| = \|\cdot\|_\infty$ in (5), where for matrix A with rows $a_1, \ldots, a_n$,

$$\|A\|_\infty = \max_{i \in \{1,\ldots,n\}} \|a_i\|_1$$

(Natural choice, since $\|\cdot\|_{TV} = \|\cdot\|_1/2$ and each row of Π is a distribution)

Modified dimensionality measure

For analytical/algorithmic reasons, we let $K \subset V$ and upper bound $\Delta(\epsilon)$ as

$$\begin{aligned} \Delta(K, \epsilon) &= |K| + \left| \left\{ v \notin K : \min_{\mu_v(k)} \left\| \pi_v - \sum_{k \in K} \mu_v(k) \pi_k \right\|_1 > \epsilon \right\} \right| \\ \mathcal{D}(k, \epsilon) &= K \cup \left\{ v \notin K : \min_{\mu_v(k)} \left\| \pi_v - \sum_{k \in K} \mu_v(k) \pi_k \right\|_1 > \epsilon \right\} \end{aligned} \tag{6}$$

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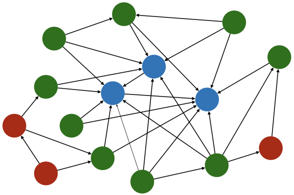
- Think of K as hub nodes (located "centrally" in graph)
- Will argue that for most non-hubs, PPR close to linear combo of hub PPR
- Second term in (6) accounts for other non-hubs (typically "far" from hubs)



Brief discussion

Modified dimensionality measure

- Collecting π_v for $v \in \mathcal{D}(K, \epsilon)$ and weights $\boldsymbol{\mu}_v$ for $v \in V$ gives a factorization of Π as H, W
  What are the dimensions? Can this be generated fast?

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Would like K to be easily identified too
Can we take nodes that will be visited “first” by random walks?

Graph model

$\Delta(K, \epsilon)$ highly dependent on local graph structure – hard to bound in general

[1] "Nice" $=$ well-approximated by a certain branching process, e.g. Chen, Olvera-Cravioto 2013; Chen, Litvak, Olvera-Cravioto 2017

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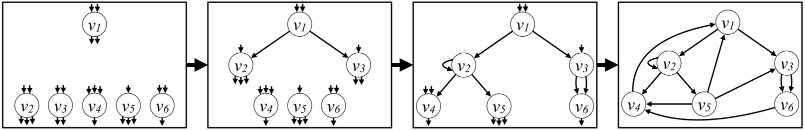
DCM construction:

1. Realize degree sequence $\{d_{out}(v), d_{in}(v)\}_{v \in V}$
2. Attach $d_{out}(v)$ ($d_{in}(v)$, resp.) outgoing (incoming, resp.) half-edges to v
3. Randomly pair half-edges to form edges via breadth-first-search



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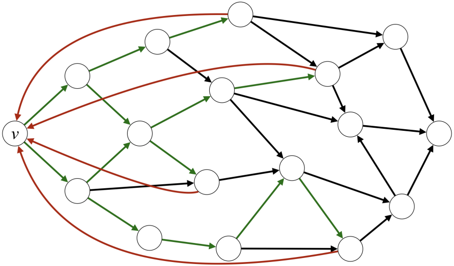
Jump probability and dimensionality

Choice of $\alpha = \mathbb{P}(\text{jump to } v)$ impacts dimensionality:

- $\alpha \approx 0 \Rightarrow \pi_v \approx$ random walk stationary distribution $\Rightarrow \Delta(K, \epsilon) \approx 1$
- $\alpha \approx 1 \Rightarrow \pi_v \approx$ point mass on $v \Rightarrow \Delta(K, \epsilon) \approx n$



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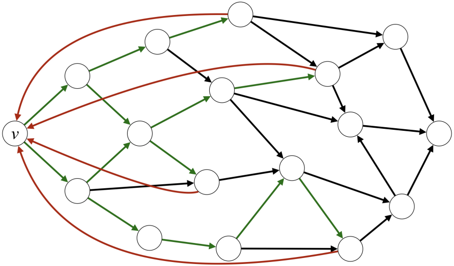

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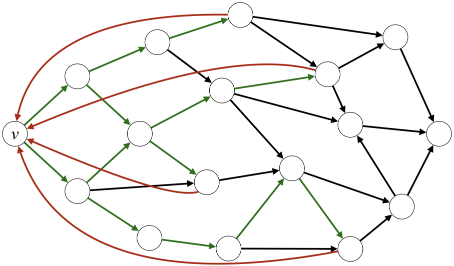
How to make this precise?

Namely, for a sequence $\{G_n\}_{n \in \mathbb{N}}$ of DCMs, how should $\alpha = \alpha_n$ scale with n?



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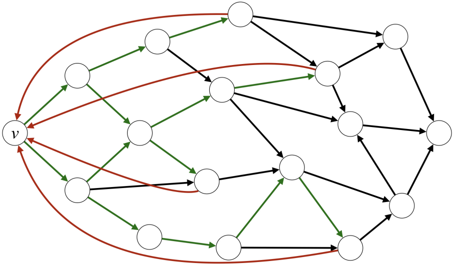

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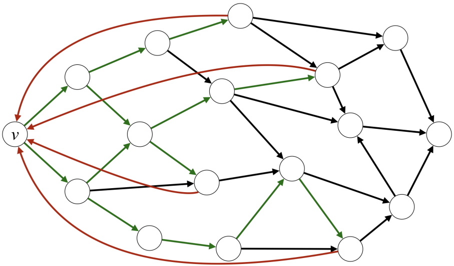

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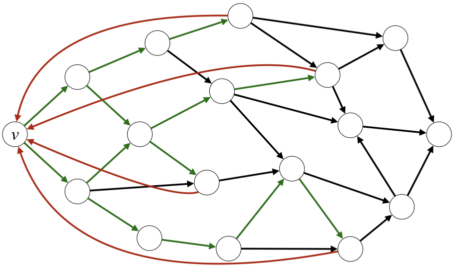

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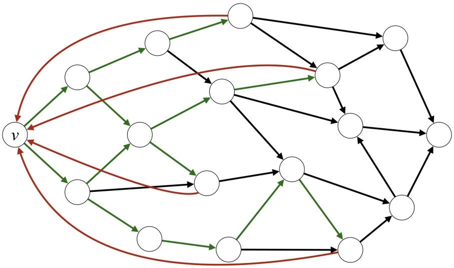

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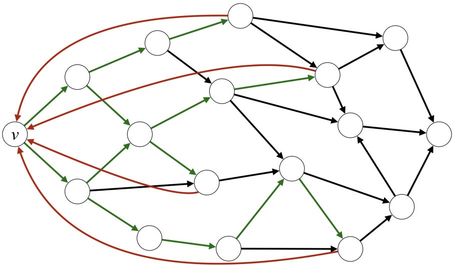
Hence, we set $\alpha_n = \Theta(1/\log n)$ (just outside the trivial regime)



Brief discussion

Random walk and PPR properties

1. If $\alpha_n =$ constant, then fixed PPR set around any node is constant-sized.
2. If $\alpha_n = \Theta(1/\log n)$, then fixed PPR set around any node increases as n^γ.
3. Scaling also related to the Cheeger number/isoperimetric number of graph family.
4. Recent results of Caputo and Quattroppani also suggest that dimension will be degenerate for any other scaling.
5. Bordenave, Caputo, Salez 2018: Random walk stationary distribution unknown but close in a strong-sense to normalized in-degree distribution.
6. Using high in-degree nodes as hubs will be good.
7. Other choices: high (Global) PageRank but needs a computation.

Main result

Main result concerns sequence of DCMs $\{G_n\}_{n \in \mathbb{N}}$, where G_n has n nodes

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Our main result says $\Delta_n(K_n,\epsilon) = o(n)$ with high probability as $n\to\infty$:

Theorem

Assume degree sequence satisfies certain assumptions (details to come), and assume $\alpha_n = \Theta(1/\log n)$. Then for any $\epsilon > 0$, some $c_\epsilon \in (0,1)$, and any $C > 0$, all independent of n,

$$\lim_{n\to\infty} \mathbb{P}\left(\Delta_n(K_n,\epsilon) > Cn^{c_\epsilon}\right) = 0.$$

Proof of main result

Main result follows almost immediately from key lemma:

Lemma

Under assumptions of theorem, we have for $s \sim V$ uniformly and some $\tilde{c}_\epsilon > 0$,

$$\mathbb{P}\left(\underbrace{\min_{\mu_s(k)} \left\| \pi_s - \sum_{k \in K} \mu_s(k) \pi_k \right\|_1}_{\star} > \epsilon\right) = O\left(n^{-\tilde{c}_\epsilon}\right).$$

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Outline for proof of lemma:

1. Show $\star$ depends only on neighborhood of s **for certain** $\mu_s(k)$
2. Approximate neighborhood construction with branching process (using Chen, Litvak, Olvera-Cravioto 2017) to study $\star$ on tree
3. Recursive nature of branching process $\rightarrow$ $\star$ on tree is martingale-like $\rightarrow$ analyze similar to method of bounded differences

Choice of $\mu_v(k)$

By considering first step of PPR Markov chain, can show

$$\pi_v(w) = \underbrace{\alpha 1(w = v)}_{\text{first step is jump to } v} + \underbrace{\sum_{k:v \to k} \frac{(1-\alpha)}{|\{k : v \to k\}|} \pi_k(w)}_{\text{first step follows random walk}}$$

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For any $K \subset V$, Jeh, Widom 2003 proves decomposition of same form:

$$\pi_v(w) = \frac{\alpha 1(w \notin K)\tilde{\pi}_v(w)}{\alpha + (1-\alpha)\tilde{\pi}_v(K)} + \sum_{k \in K} \frac{\tilde{\pi}_v(k)}{\alpha + (1-\alpha)\tilde{\pi}_v(K)} \pi_k(w)$$

where $\tilde{\pi}_v$ is PPR on graph with outgoing edges from K removed

Choice of $\mu_v(k)$

By considering first step of PPR Markov chain, can show

$$\pi_v(w) = \underbrace{\alpha 1(w = v)}_{\text{first step is jump to } v} + \underbrace{\sum_{k:v \to k} \frac{(1-\alpha)}{|\{k : v \to k\}|} \pi_k(w)}_{\text{first step follows random walk}}$$

For any $K \subset V$, Jeh, Widom 2003 proves decomposition of same form:

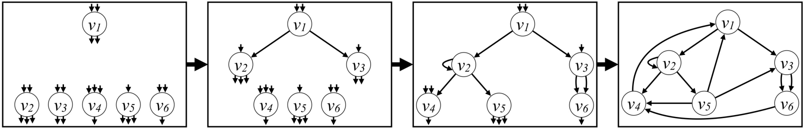
$$\pi_v(w) = \frac{\alpha 1(w \notin K)\tilde{\pi}_v(w)}{\alpha + (1-\alpha)\tilde{\pi}_v(K)} + \sum_{k \in K} \frac{\tilde{\pi}_v(k)}{\alpha + (1-\alpha)\tilde{\pi}_v(K)} \pi_k(w)$$

where $\tilde{\pi}_v$ is PPR on graph with outgoing edges from K removed

In proof (and later, in algorithm), we let $\mu_v(k) = \frac{\tilde{\pi}_v(k)}{\alpha+(1-\alpha)\tilde{\pi}_v(K)}$

Assumptions (1/2)

Recall: DCM randomly pairs edges from degree sequence $\{d_{out}(v), d_{in}(v)\}_{v \in V}$



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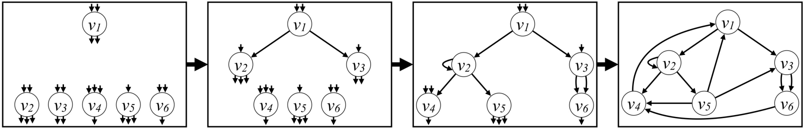

We assume $\{d_{out}(v), d_{in}(v)\}_{v \in V}$ satisfies two properties with high probability

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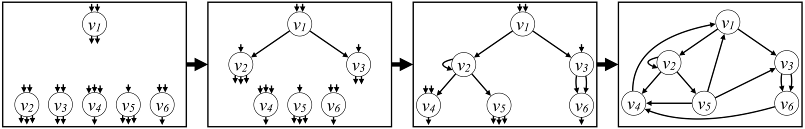

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$\Rightarrow$ Needed for branching process approximation; possible artifact of analysis

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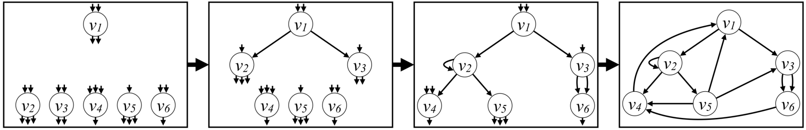

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Property 1: $\{d_{out}(v), d_{in}(v)\}_{v \in V}$ is sparse (e.g. $O(n)$ total edges)

$\Rightarrow$ Needed for branching process approximation; possible artifact of analysis

Property 2: $|K| = o(n)$ but K contains non-vanishing fraction of edges, i.e.

$$\frac{\sum_{k \in K} d_{in}(k)}{\sum_{v \in V} d_{in}(v)} \xrightarrow[n \to \infty]{} p > 0$$

$\Rightarrow$ We believe this assumption is **fundamentally necessary**

Assumptions (2/2)

Recall key property:

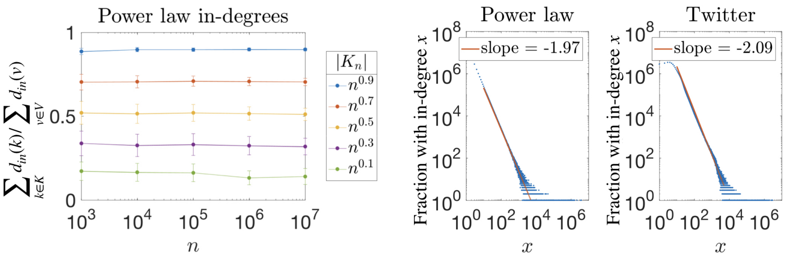
$$|K| = o(n), \quad \frac{\sum_{k \in K} d_{in}(k)}{\sum_{v \in V} d_{in}(v)} \xrightarrow[n \to \infty]{} p > 0 \tag{7}$$

Assumptions (2/2)

Recall key property:

$$|K| = o(n), \quad \frac{\sum_{k \in K} d_{in}(k)}{\sum_{v \in V} d_{in}(v)} \xrightarrow[n \to \infty]{} p > 0 \qquad (7)$$

Empirically holds if $d_{in}(v)$ follow power law, common model for e.g. Twitter



Geometric interpretation of theorem

Theorem says for most $v \notin K$ and some $\mu_v(k) \geq 0$,

$$\pi_v \approx \sum_{k \in K} \mu_v(k) \pi_k$$

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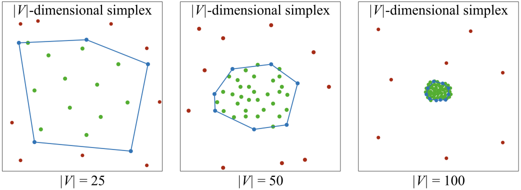
$$\pi_v \approx \text{convex combination of } \{\pi_k\}_{k \in K}$$

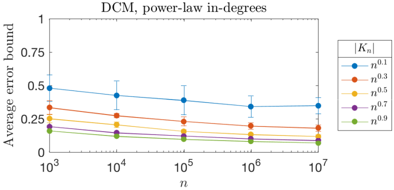
$\Rightarrow$ Most of $\{\pi_v\}_{v \notin K}$ lie near convex hull of $\{\pi_k\}_{k \in K}$, which shrinks relative to $|V|$-dimensional simplex (a few $\{\pi_v\}_{v \notin K}$ can be far away)



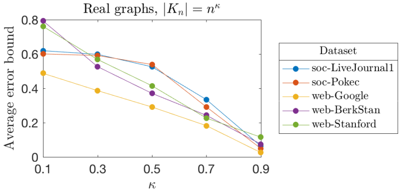
Empirical results (1/2)

Compute bound on $\|\pi_v - \sum_{k \in K} \mu_v(k) \pi_k\|_1$, averaged across $v \notin K$

Set K = nodes of highest in-degree, $\alpha_n = 1/\log n$



For DCM with power law in-degrees, average error decays as n grows (despite $|K|/n$ decaying too)



For variety of real graphs, average error decays as κ grows when $K = n^\kappa$ nodes of highest in-degree

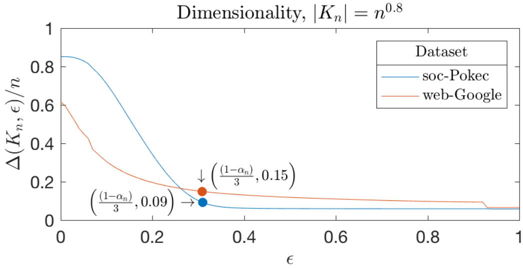
Empirical results (2/2)

Bound $\Delta(K, \epsilon)$ for two real graphs (social network, partial web crawl)

K and α_n chosen as in previous slide

For soc-Pokec, $\Delta(K, \epsilon) = 0.09n$ when $\epsilon = \frac{1-\alpha_n}{3}$; similar for web-Google[2]

Thus, while theorem doesn't apply, $\Delta(K, \epsilon)$ small relative to n for reasonable ϵ

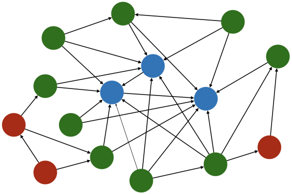


[2] Can show worst-case error is $1 - \alpha_n$, so this ϵ reduces worst-case by factor of 3

Baseline algorithm (Jeh, Widom 2003)

Jeh, Widom 2003 proposes (but doesn't analyze!) the following:

1. Choose "hub" nodes, estimate PPR vectors directly
2. For other nodes, estimate PPR as linear combo of hub PPR[3]



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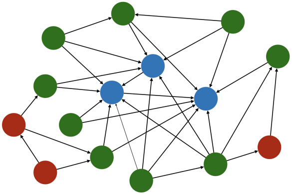

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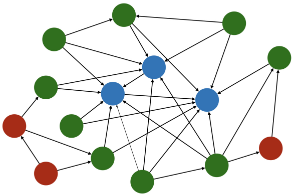
Thus, we improve Jeh, Widom 2003, but questions remain:

- Can we guarantee accuracy *all* nodes?
- Can we estimate hub PPR, and non-hub linear combo weights, with **provably good performance?** (good heuristics such as Global PageRank in Jeh, Widom 2003)



[3] Using decomposition shown previously

Improving accuracy of baseline scheme

Baseline scheme: for $v \notin K$, π_v estimated as

$$\hat{\pi}_v = \sum_{k \in K} \mu_v(k) \pi_k$$

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We show (for a certain function f)

$$\|\pi_v - \hat{\pi}_v\|_1 < \epsilon \Leftrightarrow \sum_{k \in K} \mu_v(k) > f(\epsilon)$$

Intuitively, small error $\Leftrightarrow$ v is "close" to K in graph

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Key point: $\sum_{k \in K} \mu_v(k)$ is (approximately) known at runtime!

$\Rightarrow$ If $\sum_{k \in K} \mu_v(k) < f(\epsilon)$, estimate π_v directly

Estimating PPR and linear combo weights (1/2)

Recall: π_v = stationary distribution of chain with transition matrix

$$P_v = \underbrace{(1-\alpha)P}_{\text{Random walk}} + \underbrace{\alpha 1_n e_v^T}_{\text{Jump to } v}$$

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$$\Pi = \alpha(I_n - (1-\alpha)P)^{-1} = \alpha \sum_{i=0}^{\infty} (1-\alpha)^i P^i$$

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Suggests power iteration: choose i^* large and compute

$$\alpha \sum_{i=0}^{i^*} (1-\alpha)^i P^i \approx \Pi$$

Estimating PPR and linear combo weights (2/2)

Power iteration traverses all paths of length $\leq i^*$

Directed Laplacian variants:

- Set $i^* = \Theta(\log(n))$
- Modify power method so that dense matrices do not arise

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Forward DP (Andersen, Chung, Lang 2006):

- Given v, traverses "important" paths out of v; estimates v-th row of Π
- **Can use to estimate PPR vectors directly**

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Forward DP (Andersen, Chung, Lang 2006):

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- **Can use to estimate PPR vectors directly**

Backward DP (Andersen et al. 2008):

- Given v, traverses "important" paths into v; estimates v-th column of Π
- **Can use (modified version) to estimate linear combo weights**

Putting it all together

Our scheme estimates π_v . . .

- . . . by forward DP, if $v \in K$
- . . . by forward DP, if $v \notin K$ and linear combo determined to be inaccurate
- . . . as linear combo, if $v \notin K$ and linear combo determined to be accurate

Forward DP provably accurate; thus, all estimates are accurate

[4] Assuming $|E| = O(n), \alpha = \Theta(1/\log n)$

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Forward DP provably accurate; thus, all estimates are accurate

Complexity dominated by number runs of forward DP

- By design, forward DP is run $\Delta(K, \epsilon)$ times
- Each run has $O(n \log n)$ complexity (by Andersen, Chung, Lang 2006)[4]

Overall complexity is $O(\Delta(K, \epsilon) n \log n) = o(n^2)$ (when theorem applies)

[4] Assuming $|E| = O(n), \alpha = \Theta(1/\log n)$

Comparison to existing algorithms

Best existing approach: run forward or backward DP $\forall\ v$

- l_1 accuracy guarantee, $O(n^2 \log n)$ complexity
- Ignores structure/dependencies across rows of Π!
- Our scheme accounts for structure, thus reduces complexity

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Another noteworthy work: Lofgren, Banerjee, Goel 2016

- Estimates single entry of Π via DP + MCMC, complexity $O(\sqrt{n} \log n)$
- Hence, $O(n^{2.5} \log n)$ to estimate Π; ignores dependencies across entries
- Again, accounting for structure allows us to reduce complexity

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Connections to other problems:

- Non-negative matrix factorization: Unknown $n \times n$ Π split into non-negative factors $n \times \tilde{k}$ and $\tilde{k} \times n$ factors in $o(n^2)$ time
  Related work Sen et al. 2016 is in a different norm.

Thanks!

Thanks for your attention

Paper appeared in ACM SIGMETRICS 2019

Questions?

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