Personalized PageRank dimensionality and algorithmic implications

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Motivation

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Social networks

Internet

Power grid
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Questions of interest in this talk:
- Which nodes are most important/influential, globally and locally?
- Which nodes are similar/relevant to a given node?
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![Social networks](image1)
![Internet](image2)
![Power grid](image3)

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- Which nodes are most important/influential, globally and locally?
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One metric used to answer these questions: Personalized PageRank (PPR)
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- W.p. $(1 - \alpha)$, sample $X_{t+1}^v$ from out-neighbors of $X_t^v$ (random walk)
- W.p. $\alpha$, set $X_{t+1}^v = v$ (jump to $v$)
PPR definition

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Stationary distribution $\pi_v = \{\pi_v(w)\}_{w \in V}$ called PPR vector
Given directed graph $G = (V, E)$, let $v \in V$ and $\alpha \in (0, 1)$

Define Markov chain $\{X^v_t\}_{t \in \mathbb{N}}$ as follows: given $X^v_t$,

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Stationary distribution $\pi_v = \{\pi_v(w)\}_{w \in V}$ called **PPR vector**

Matrix $\Pi = \{\pi_v\}_{v \in V}$ called **PPR matrix**
PPR interpretation

\[ \pi_v(w) \text{ large when } w \text{ frequently visited on short walks from } v \]

\[ \Rightarrow \text{ Interpret } \pi_v(w) \text{ as measure of } w \text{'s importance/relevance to } v \]
PPR interpretation

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(Global) PageRank and PPR
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(Global) PageRank and PPR

Proposed to rank websites (Page et al. 1999); many uses since
- Recommendation (Baluja et al. 2008; Gupta et al. 2013)
- Bioinformatics (Morrison et al. 2005; Freschi 2007)
- Community detection (Andersen, Chung, Lang 2006, Kloumann, Ugander, Kleinberg 2017)
- Graph similarity (Koutra, Vogelstein, Faloutsos 2013)
- ...
Definitions

Directed graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $m = |E|$

Adjacency $A$, diagonal out-degree $D$, $P = D^{-1}A$ (row stochastic)

PPR: Perron-Frobenius eigenvector $\pi_\sigma$ of non-negative matrix

$$P_\sigma = (1 - \alpha)P + \alpha 1_n \sigma^T$$  \hspace{1cm} (1)

where $\alpha \in (0, 1)$, $\sigma \in \mathbb{R}_+^n$ s.t. $\sum_{v \in V} \sigma(v) = 1$ (distribution on $V$)

When $\sigma = e_s$ (1 in $s$-th position, 0 elsewhere), denote as $\pi_s$

(Global) PageRank: $\sigma = \frac{1}{n} 1_n$
Computation of PPR

- Linear algebraic method
  - Non-negative matrix, so Perron-Frobenius theorem
  - Power method variants, $O(n^2)$ for each source
  - Directed Laplacian variants, almost linear time; Miller, Spielman, Teng, Peng.

- Probabilistic method
  - $P$ is the transition kernel of simple random walk on $G$
  - Use Monte Carlo (random walks) to estimate PPR
  - $O(n \log(n))$ complexity for (Global) PageRank: Avrachenkov et al. 2007, Sarma et al. 2015

- Variational method
  - View eigenvector computation as a Bellman equation
  - Use value iteration: Andersen, Chung, Lang 2006, Andersen et al. 2008

- Hybrid schemes
  - Monte Carlo + Variational method
  - For a single entry of $\Pi$ - Lofgren, Banerjee, Goel 2016
Key properties

$P_\sigma$ is a Doeblin chain: Athreya, Stenflo 2003

$$\pi_s(t) = \mathbb{P}[\text{random walk from } s \text{ of length } \sim \text{geom}(\alpha) \text{ ends at } t] \quad (2)$$

⇒ Can sample from $\pi_s$ using random walks

To estimate $\pi_\sigma$, suffices to estimate $\pi_s$, because

$$\pi_\sigma = \sum_{s \in V} \sigma(s) \pi_s \quad (3)$$

Renewal reward interpretation: $\pi_s(t)$ importance of $t$ for $s$, as

$$\pi_s(t) \propto \mathbb{E}[\text{number of visits to } t \text{ on } \text{geom}(\alpha)-\text{length walk from } s] \quad (4)$$
Using Perron-Frobenius theorem, can show \( \text{rank}(\Pi) = |V| = n \)
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However, PPR exhibits transitive structure
- \( \pi_{v_1}(v_2), \pi_{v_2}(v_3) \) large \( \Rightarrow \pi_{v_1}(v_3) \) large ("friend of my friend is my friend")
- Suggests \( \Pi \) has small "effective dimension"
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Also, for many real-world graphs $G = (V, E), |E| = O(n)$
- Suggests $G$ is $O(n)$-dimensional, but $\Pi$ (derived from $G$) is $n^2$-dimensional
- Is this gap actually present?
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Outline of talk:
1. How to quantify effective dimension of \( \Pi \)?
2. Can we bound this measure of dimensionality?
3. If bound "small", can we leverage it algorithmically?
Quantifying PPR dimensionality

Natural measure of effective dimension of $\Pi$:

$$\Delta(\epsilon) = \min_{\hat{\Pi}} \text{rank}(\hat{\Pi}) \text{ s.t. } \|\Pi - \hat{\Pi}\| < \epsilon$$  \hspace{1cm} (5)

Intuitively, $\Pi$ low dimensional if close to low-rank matrix
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We take \( \| \cdot \| = \| \cdot \|_\infty \) in (5), where for matrix \( A \) with rows \( a_1, \ldots, a_n \),

\[
\| A \|_\infty = \max_{i \in \{1, \ldots, n\}} \| a_i \|_1
\]

(Natural choice, since \( \| \cdot \|_{TV} = \| \cdot \|_1/2 \) and each row of \( \Pi \) is a distribution)
Modified dimensionality measure

For analytical/algorithmic reasons, we let $K \subset V$ and upper bound $\Delta(\epsilon)$ as

\[
\Delta(K, \epsilon) = |K| + \left| \left\{ v \notin K : \min_{\mu_v(k)} \left\| \pi_v - \sum_{k \in K} \mu_v(k) \pi_k \right\|_1 > \epsilon \right\} \right|
\]

\[
D(k, \epsilon) = K \cup \left\{ v \notin K : \min_{\mu_v(k)} \left\| \pi_v - \sum_{k \in K} \mu_v(k) \pi_k \right\|_1 > \epsilon \right\}
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- Think of $K$ as hub nodes (located “centrally” in graph)
- Will argue that for most non-hubs, PPR close to linear combo of hub PPR
- Second term in (6) accounts for other non-hubs (typically “far” from hubs)
Brief discussion

Modified dimensionality measure

- Collecting $\pi_v$ for $v \in D(K, \epsilon)$ and weights $\mu_v$ for $v \in V$ gives a factorization of $\Pi$ as $H, W$

What are the dimensions? Can this be generated fast?
Modified dimensionality measure

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  What are the dimensions? Can this be generated fast?

Would like $K$ to be easily identified too

Can we take nodes that will be visited “first” by random walks?
Graph model

\[ \Delta(K, \epsilon) \text{ highly dependent on local graph structure} \text{ – hard to bound in general} \]

---

1 “Nice” = well-approximated by a certain branching process, e.g. Chen, Olvera-Cravioto 2013; Chen, Litvak, Olvera-Cravioto 2017
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We analyze directed configuration model (DCM) due to “nice” local structure\(^1\)

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We analyze directed configuration model (DCM) due to “nice” local structure

DCM construction:

1. Realize degree sequence \( \{d_{out}(v), d_{in}(v)\}_{v \in V} \)
2. Attach \( d_{out}(v) \) (\( d_{in}(v) \), resp.) outgoing (incoming, resp.) half-edges to \( v \)
3. Randomly pair half-edges to form edges via breadth-first-search

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Jump probability and dimensionality

Choice of $\alpha = \mathbb{P}(\text{jump to } v)$ impacts dimensionality:

- $\alpha \approx 0 \Rightarrow \pi_v \approx \text{random walk stationary distribution} \Rightarrow \Delta(K, \epsilon) \approx 1$
- $\alpha \approx 1 \Rightarrow \pi_v \approx \text{point mass on } v \Rightarrow \Delta(K, \epsilon) \approx n$
Jump probability and dimensionality

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How to make this precise?
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How to make this precise?

Namely, for a sequence $\{G_n\}_{n \in \mathbb{N}}$ of DCMs, how should $\alpha = \alpha_n$ scale with $n$?
Jump probability and mixing times

Suppose $\alpha_n \log n \to 0$, e.g. $\alpha_n = 1/(\log n)^2$
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Mixing occurs before jump to $v$! Allows us to show $\Delta(K, \epsilon) = 1$ with high prob.
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Hence, we set $\alpha_n = \Theta(1/\log n)$ (just outside the trivial regime)
Brief discussion

Random walk and PPR properties

1. If \( \alpha_n = \text{constant} \), then fixed PPR set around any node is constant-sized.
2. If \( \alpha_n = \Theta(1/\log n) \), then fixed PPR set around any node increases as \( n^\gamma \).
3. Scaling also related to the Cheeger number/isoperimetric number of graph family.
4. Recent results of Caputo and Quattropani also suggest that dimension will be degenerate for any other scaling.
5. Bordenave, Caputo, Salez 2018: Random walk stationary distribution unknown but close in a strong-sense to normalized in-degree distribution.
6. Using high in-degree nodes as hubs will be good.
7. Other choices: high (Global) PageRank but needs a computation.
Main result

Main result concerns sequence of DCMs \( \{G_n\}_{n \in \mathbb{N}} \), where \( G_n \) has \( n \) nodes.
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From \( G_n \), define \( \Delta_n(K_n, \epsilon) \) for specific \( K_n \) (random variable, as \( G_n \) is random)
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Our main result says \( \Delta_n(K_n, \epsilon) = o(n) \) with high probability as \( n \to \infty \):

**Theorem**

*Assume degree sequence satisfies certain assumptions (details to come), and assume \( \alpha_n = \Theta(1/\log n) \). Then for any \( \epsilon > 0 \), some \( c_\epsilon \in (0, 1) \), and any \( C > 0 \), all independent of \( n \),

\[
\lim_{n \to \infty} \mathbb{P}(\Delta_n(K_n, \epsilon) > Cn^{c_\epsilon}) = 0.
\]
Proof of main result

Main result follows almost immediately from key lemma:

Lemma

*Under assumptions of theorem, we have for $s \sim V$ uniformly and some $\tilde{c}_\epsilon > 0$,*

$$
\Pr \left( \min_{\mu_s(k)} \left\| \pi_s - \sum_{k \in K} \mu_s(k) \pi_k \right\|_1 > \epsilon \right) = O \left( n^{-\tilde{c}_\epsilon} \right).
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Outline for proof of lemma:

1. Show $\star$ depends only on neighborhood of $s$ for certain $\mu_s(k)$
2. Approximate neighborhood construction with branching process (using Chen, Litvak, Olvera-Cravioto 2017) to study $\star$ on tree
3. Recursive nature of branching process $\rightarrow \star$ on tree is martingale-like $\rightarrow$ analyze similar to method of bounded differences
Choice of $\mu_v(k)$

By considering first step of PPR Markov chain, can show

$$\pi_v(w) = \alpha 1(w = v) + \sum_{k: v \rightarrow k} \frac{(1 - \alpha)}{|\{k : v \rightarrow k\}|} \pi_k(w)$$

where $\tilde{\pi}_v$ is PPR on graph with outgoing edges from $K$ removed
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first step is jump to $v$

first step follows random walk

For any $K \subset V$, Jeh, Widom 2003 proves decomposition of same form:

$$\pi_v(w) = \frac{\alpha 1(w \notin K) \tilde{\pi}_v(w)}{\alpha + (1 - \alpha) \tilde{\pi}_v(K)} + \sum_{k \in K} \frac{\tilde{\pi}_v(k)}{\alpha + (1 - \alpha) \tilde{\pi}_v(K)} \pi_k(w)$$

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In proof (and later, in algorithm), we let $\mu_v(k) = \frac{\tilde{\pi}_v(k)}{\alpha + (1 - \alpha) \tilde{\pi}_v(K)}$
Recall: DCM randomly pairs edges from degree sequence \( \{d_{out}(v), d_{in}(v)\}_{v \in V} \)

Assumptions (1/2)

We assume \( \{d_{out}(v), d_{in}(v)\}_{v \in V} \) satisfies two properties with high probability:

**Property 1:**
\( \{d_{out}(v), d_{in}(v)\}_{v \in V} \) is sparse (e.g. \( O(n) \) total edges) → Needed for branching process approximation; possible artifact of analysis

**Property 2:**
\( |K| = o(n) \) but \( K \) contains non-vanishing fraction of edges, i.e.
\[
\sum_{k \in K} d_{in}(k) \rightarrow \sum_{v \in V} d_{in}(v) - \rightarrow n \rightarrow \infty \quad p > 0
\]
We believe this assumption is fundamentally necessary.
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Recall key property:

$$|K| = o(n), \quad \frac{\sum_{k \in K} d_{in}(k)}{\sum_{v \in V} d_{in}(v)} \xrightarrow{n \to \infty} p > 0$$  (7)
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Empirically holds if \( d_{in}(v) \) follow power law, common model for e.g. Twitter
Geometric interpretation of theorem

Theorem says for most $v \notin K$ and some $\mu_v(k) \geq 0$,

$$
\pi_v \approx \sum_{k \in K} \mu_v(k) \pi_k 
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Geometric interpretation of theorem

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When \( |V| \) large, we also show \( \sum_{k \in K} \mu_v(k) \approx 1 \), so for most \( v \notin K \),

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\]

\(\Rightarrow\) Most of \(\{\pi_v\}_{v \notin K}\) lie near convex hull of \(\{\pi_k\}_{k \in K}\), which shrinks relative to \(|V|\)-dimensional simplex (a few \(\{\pi_v\}_{v \notin K}\) can be far away)
Compute bound on $\|\pi_v - \sum_{k \in K} \mu_v(k)\pi_k\|_1$, averaged across $v \notin K$

Set $K =$ nodes of highest in-degree, $\alpha_n = 1/\log n$

For DCM with power law in-degrees, average error decays as $n$ grows (despite $|K|/n$ decaying too)

For variety of real graphs, average error decays as $\kappa$ grows when $K = n^\kappa$ nodes of highest in-degree
Empirical results (2/2)

Bound $\Delta(K, \epsilon)$ for two real graphs (social network, partial web crawl)

$K$ and $\alpha_n$ chosen as in previous slide

For soc-Pokec, $\Delta(K, \epsilon) = 0.09n$ when $\epsilon = \frac{1-\alpha_n}{3}$; similar for web-Google\(^2\)

Thus, while theorem doesn’t apply, $\Delta(K, \epsilon)$ small relative to $n$ for reasonable $\epsilon$

\(^2\)Can show worst-case error is $1 - \alpha_n$, so this $\epsilon$ reduces worst-case by factor of 3
Baseline algorithm (Jeh, Widom 2003)

Jeh, Widom 2003 proposes (but doesn't analyze!) the following:

1. Choose “hub” nodes, estimate PPR vectors directly
2. For other nodes, estimate PPR as linear combo of hub PPR$^3$

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$^3$Using decomposition shown previously
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Thus, we improve Jeh, Widom 2003, but questions remain:

- Can we guarantee accuracy all nodes?
- Can we estimate hub PPR, and non-hub linear combo weights, with provably good performance? (good heuristics such as Global PageRank in Jeh, Widom 2003)

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3 Using decomposition shown previously
Improving accuracy of baseline scheme

Baseline scheme: for $v \notin K$, $\pi_v$ estimated as

$$\hat{\pi}_v = \sum_{k \in K} \mu_v(k) \pi_k$$

where $\mu_v(k)$ from linear decomposition shown previously
Improving accuracy of baseline scheme

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We show (for a certain function $f$)

$$\|\pi_v - \hat{\pi}_v\|_1 < \epsilon \Leftrightarrow \sum_{k \in K} \mu_v(k) > f(\epsilon)$$

Intuitively, small error $\Leftrightarrow v$ is “close” to $K$ in graph
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Key point: $\sum_{k \in K} \mu_v(k)$ is (approximately) known at runtime!

$\Rightarrow$ If $\sum_{k \in K} \mu_v(k) < f(\epsilon)$, estimate $\pi_v$ directly
Estimating PPR and linear combo weights (1/2)

Recall: $\pi_v = \text{stationary distribution of chain with transition matrix}$

$$P_v = (1 - \alpha) P + \alpha 1_v e_v^T$$

Random walk Jump to $v$
Recall: $\pi_v = \text{stationary distribution of chain with transition matrix}$

\[ P_v = (1 - \alpha)P + \alpha \mathbf{1}_n \mathbf{e}_v^T \]

\[ \text{Random walk} \quad \text{Jump to } v \]

Solving $\pi_v = \pi_v P_v$ yields

\[ \pi_v = \alpha \mathbf{e}_v^T (I_n - (1 - \alpha)P)^{-1} \]
Recall: $\pi_v = \text{stationary distribution of chain with transition matrix}$

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Since $\pi_v$ is $v$-th row of $\Pi$,

$$\Pi = \alpha (I_n - (1 - \alpha)P)^{-1} = \alpha \sum_{i=0}^{\infty} (1 - \alpha)^i P^i$$
Recall: \( \pi_v \) = stationary distribution of chain with transition matrix

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Random walk Jump to \( v \)

Solving \( \pi_v = \pi_v P_v \) yields

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\[
\Pi = \alpha (I_n - (1 - \alpha)P)^{-1} = \alpha \sum_{i=0}^{\infty} (1 - \alpha)^i P^i
\]

Suggests power iteration: choose \( i^* \) large and compute

\[
\alpha \sum_{i=0}^{i^*} (1 - \alpha)^i P^i \approx \Pi
\]
Estimating PPR and linear combo weights (2/2)

Power iteration traverses all paths of length $\leq i^*$

Directed Laplacian variants:

- Set $i^* = \Theta(\log(n))$
- Modify power method so that dense matrices do not arise
Power iteration traverses all paths of length $\leq i^*$

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Dynamic programming (DP) variants traverse only “important” paths

Forward DP (Andersen, Chung, Lang 2006):
- Given $v$, traverses “important” paths out of $v$; estimates $v$-th row of $\Pi$
- Can use to estimate PPR vectors directly
Estimating PPR and linear combo weights (2/2)

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- Given $v$, traverses “important” paths out of $v$; estimates $v$-th row of $\Pi$
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Backward DP (Andersen et al. 2008):
- Given $v$, traverses “important” paths into $v$; estimates $v$-th column of $\Pi$
- Can use (modified version) to estimate linear combo weights
Putting it all together

Our scheme estimates $\pi_v$ . . .
- . . . by forward DP, if $v \in K$
- . . . by forward DP, if $v \notin K$ and linear combo determined to be inaccurate
- . . . as linear combo, if $v \notin K$ and linear combo determined to be accurate

Forward DP provably accurate; thus, all estimates are accurate

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4 Assuming $|E| = O(n)$, $\alpha = \Theta(1/ \log n)$
Putting it all together

Our scheme estimates $\pi_v$ ...

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Forward DP provably accurate; thus, all estimates are accurate

Complexity dominated by number runs of forward DP

- By design, forward DP is run $\Delta(K, \epsilon)$ times
- Each run has $O(n \log n)$ complexity (by Andersen, Chung, Lang 2006)$^4$

Overall complexity is $O(\Delta(K, \epsilon) n \log n) = o(n^2)$ (when theorem applies)

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$^4$Assuming $|E| = O(n)$, $\alpha = \Theta(1/ \log n)$
Comparison to existing algorithms

Best existing approach: run forward or backward DP $\forall \nu$

- $l_1$ accuracy guarantee, $O(n^2 \log n)$ complexity
- Ignores structure/dependencies across rows of $\Pi$
- Our scheme accounts for structure, thus reduces complexity
Comparison to existing algorithms

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Another noteworthy work: Lofgren, Banerjee, Goel 2016

- Estimates single entry of $\Pi$ via DP + MCMC, complexity $O(\sqrt{n} \log n)$
- Hence, $O(n^{2.5} \log n)$ to estimate $\Pi$; ignores dependencies across entries
- Again, accounting for structure allows us to reduce complexity
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Connections to other problems:
- Non-negative matrix factorization: Unknown $n \times n$ $\Pi$ split into non-negative factors $n \times \tilde{k}$ and $\tilde{k} \times n$ factors in $o(n^2)$ time
  Related work Sen et al. 2016 is in a different norm.
Thanks for your attention

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Questions?

Andersen, Reid et al. (2008). “Local computation of PageRank contributions”. In: Internet Mathematics 5.1-2, pp. 23–45.


