Coded and Uncoded Trace Reconstruction

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Deletion channel

0 1 0 0 1 1

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0 1 0 0 1 1

↓

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↓

0 1 1

Deletion channel

0 1 0 0 1 1

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0 1 1

Other deletion patterns that lead to same output

0 1 0 0 1 1

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Other deletion patterns that lead to same output

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i.i.d. deletions: each bit deleted independently with fixed probability d

Deletion channel

0 1 0 0 1 1

↓

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↓

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Other deletion patterns that lead to same output

0 1 0 0 1 1

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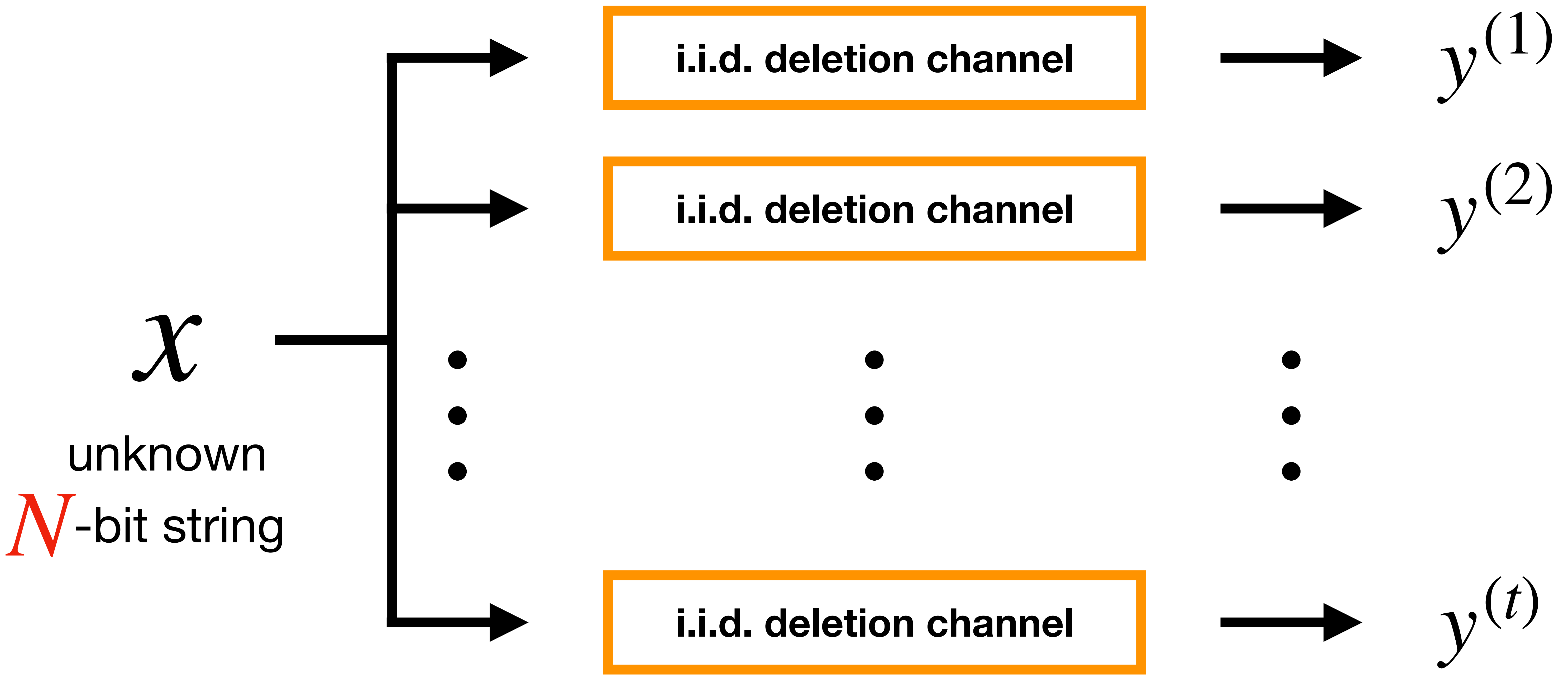
i.i.d. deletions: each bit deleted independently with fixed probability d

Many open problems! E.g., capacity is still unknown…

Trace reconstruction

[Levenshtein01, BatuKannanKhannaMcGregor04]



Goal

Reconstruct x from **traces** $y^{(1)}, \ldots, y^{(t)}$ such that:

- We succeed with high probability $1 - O(1/N)$
- We use as few traces as possible

Original motivation for trace reconstruction

[BatuKannanKhannaMcGregor04]

Ancestor DNA

mutations
(deletions, insertions, substitutions)

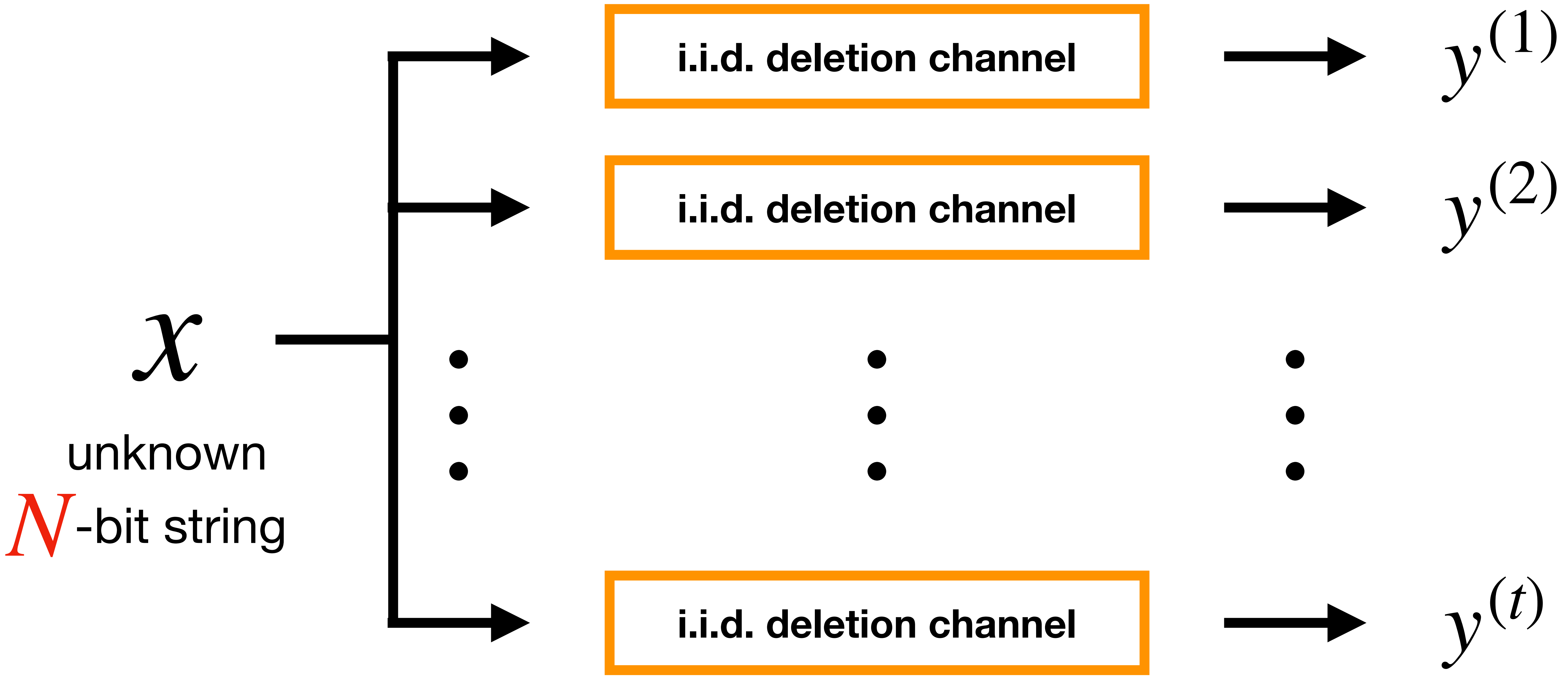
Descendent 1 DNA

Descendent 2 DNA

Descendent 3 DNA

Multiple sequence alignment: Deduce common ancestor DNA from descendants DNA.

Main settings for uncoded trace reconstruction



Worst-case trace reconstruction

*Reconstruction algorithm $\mathscr{R}$ must succeed with high probability simultaneously for **all input strings**.*

$$\forall x : \Pr_{y^{(1)},\ldots,y^{(t)} \leftarrow \mathsf{Del}(x)} [\mathscr{R}(y^{(1)}, \ldots, y^{(t)}) = x] \approx 1$$

Average-case trace reconstruction

***Average** error probability of reconstruction algorithm $\mathscr{R}$ over all input strings is small.*

$$\Pr_{x \leftarrow \{0,1\}^N, y^{(1)},\ldots,y^{(t)} \leftarrow \mathsf{Del}(x)} [\mathscr{R}(y^{(1)}, \ldots, y^{(t)}) = x] \approx 1$$

Worst-case trace reconstruction

Recall: Reconstruction algorithm must succeed for **all strings** with high probability.

Reconstruction algorithms	#traces	deletion probability
Batu, Kannan, Khanna, McGregor 2004	$N \log N$	$d = 1/N^{1/2+\epsilon}$
Holenstein, Mitzenmacher, Panigrahy, Wieder 2008	$\exp(N^{1/2})$	any constant < 1
De, O'Donnell, Servedio 2017 Nazarov, Peres 2017	$\exp(N^{1/3})$	any constant < 1

Mean-based algorithms & worst-case trace reconstruction

$x \longrightarrow$ i.i.d. deletions $\longrightarrow Y$ (padded with 0's)

$\in \{-1,1\}^N$ $\qquad\qquad$ $\in \{-1,0,1\}^N$

Mean-based algorithms & worst-case trace reconstruction

$$x \xrightarrow{\quad} \boxed{\text{i.i.d. deletions}} \xrightarrow{\quad} Y$$

$\in \{-1,1\}^N$ $\qquad\qquad$ $\in \{-1,0,1\}^N$ $\qquad$ (padded with 0's)

Mean trace: $\mu(x) = (\mathrm{E}[Y_1], \mathrm{E}[Y_2], \ldots, \mathrm{E}[Y_N])$ (real-linear function of x)

Mean-based algorithms & worst-case trace reconstruction

x ⟶ i.i.d. deletions ⟶ Y **(padded with 0's)**

$\in \{-1,1\}^N$ $\qquad$ $\in \{-1,0,1\}^N$

Mean trace: $\mu(x) = (\mathrm{E}[Y_1], \mathrm{E}[Y_2], \ldots, \mathrm{E}[Y_N])$ **(real-linear function of x)**

Mean-based algorithm:

- Compute estimate $\hat{\mu}$ of mean trace $\mu(x)$ from T traces;
- Find x such that $\mu(x)$ is closest to $\hat{\mu}$.

[DeO'DonnellServedio17, NazarovPeres17]:
There exist mean-based algorithms using $T = \exp(N^{1/3})$ traces. Moreover, **this is optimal for mean-based algorithms.**

Mean-based algorithms & complex analysis

Mean trace: $\mu(x) = (\mathrm{E}[Y_1], \mathrm{E}[Y_2], \ldots, \mathrm{E}[Y_N])$ **(real-linear function of x)**

#traces required for accurate estimate $\hat{\mu}$ is dictated by

$$\min_{x \neq x'} ||\mu(x) - \mu(x')||_1 = 2 \times \min_{x \in \{-1,0,1\}^N} ||\mu(x)||_1$$

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Deletion channel polynomial $P_{d,x}(w) = \sum_{j=1}^{N} \mu(x)_j \cdot w^j, \quad w \in \mathbb{C}$

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Bounding $\min ||\mu(x)||_1$ $\approx$ **Maximizing special polynomial over arc of complex circle**

(can use powerful complex analytic tools!)

Mean-based algorithms & Littlewood polynomials

Deletion channel polynomial $$P_{d,x}(w) = \sum_{j=1}^{N} \mu(x)_j \cdot w^j, \quad w \in \mathbb{C}$$

$$P_{d,x}(w) = (1-d)\sum_{j=1}^{N} x_j \cdot z^j \qquad z = d + (1-d)w$$

Easy to write
in terms of x

Mean-based algorithms & Littlewood polynomials

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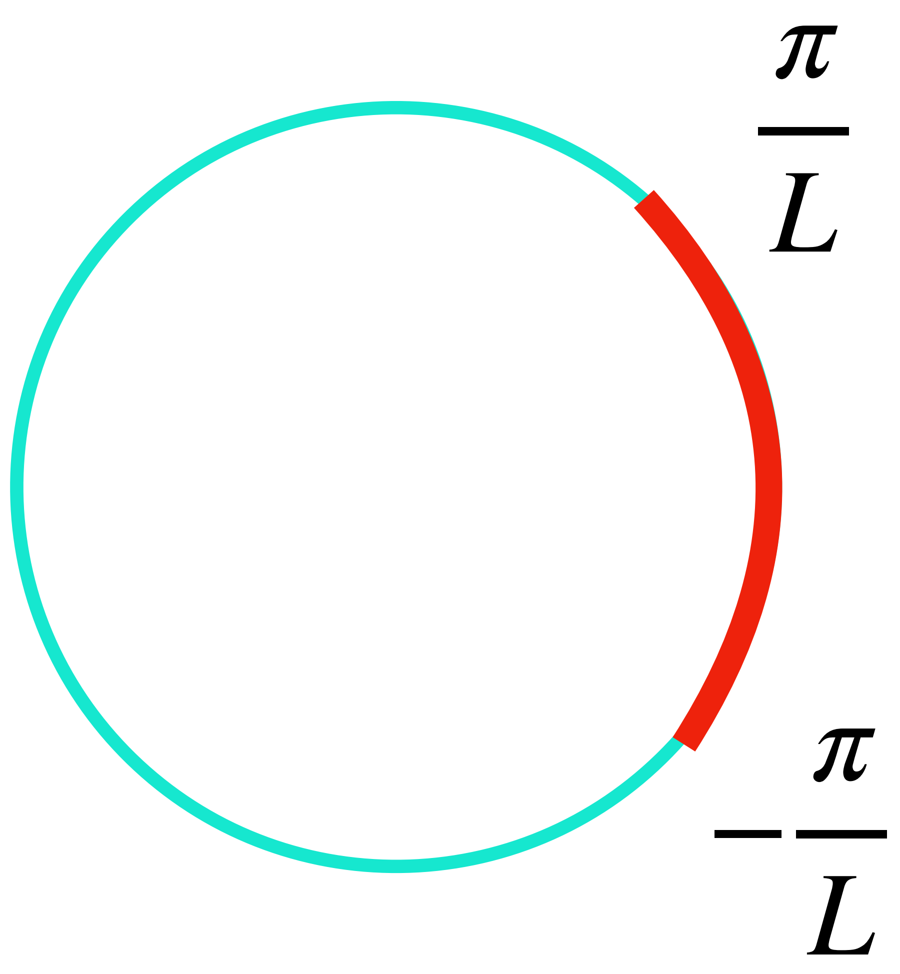

[BorweinErdélyi97]

$A(z)$ Littlewood polynomial

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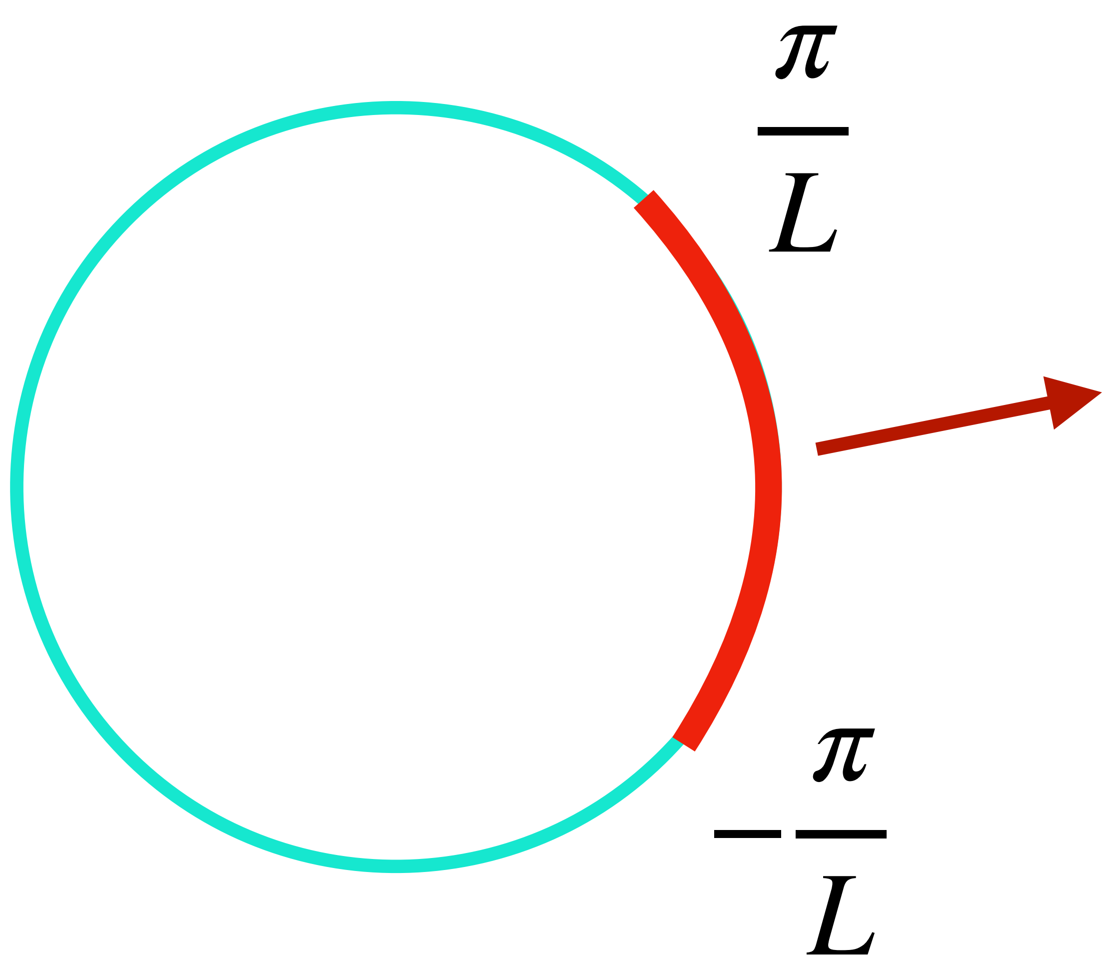

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$$\max |A(e^{i\theta})| \geq e^{-cL}$$

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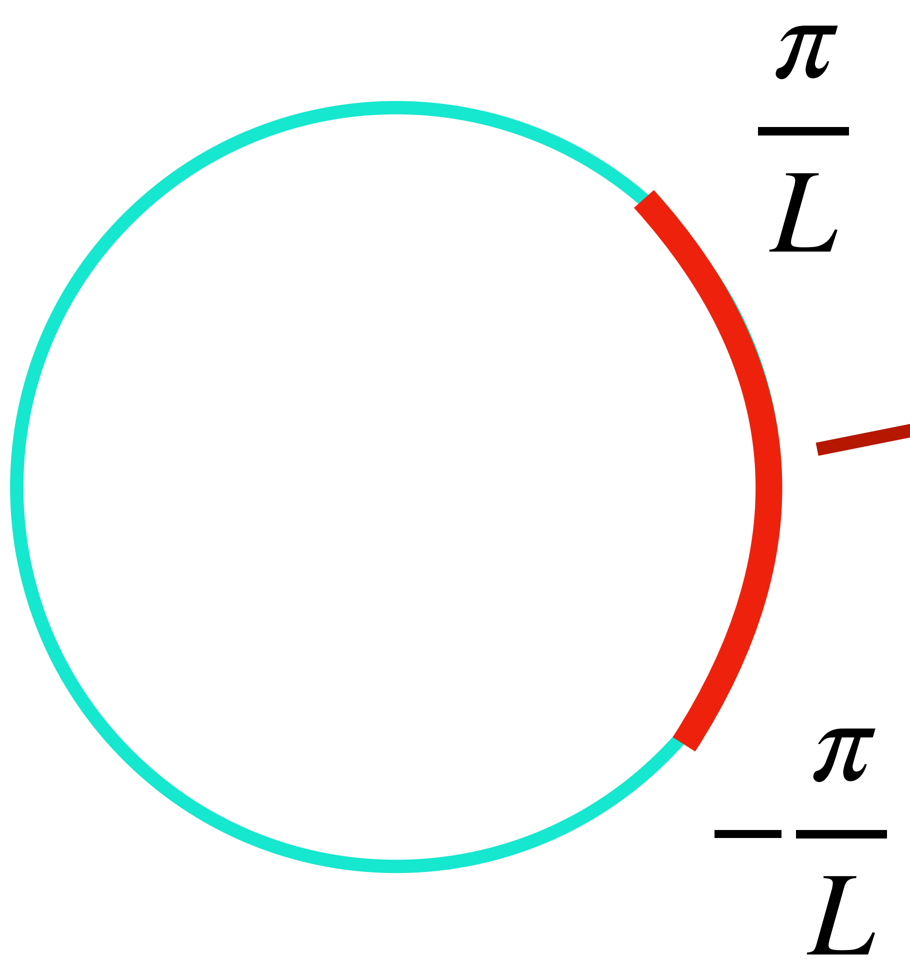

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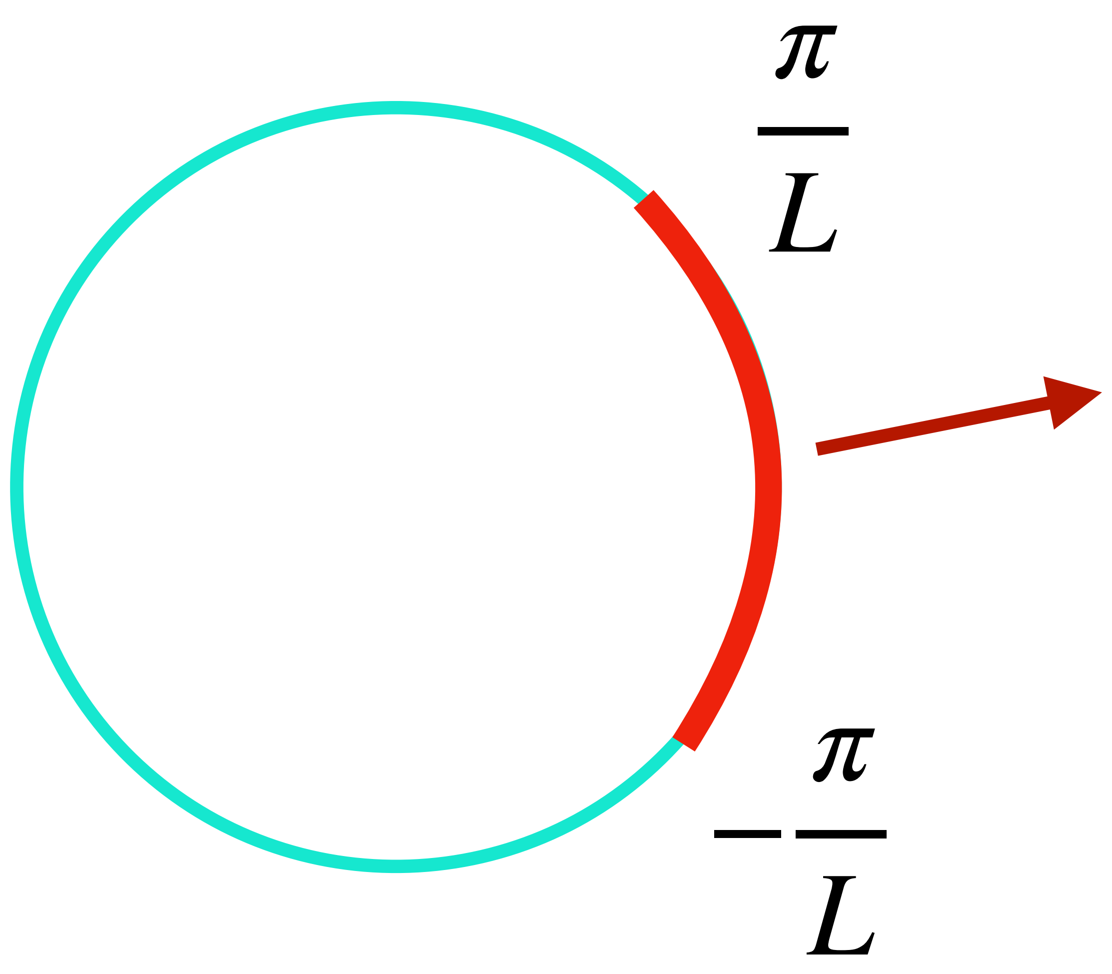

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$$\sum_j |\mu_j(x)| \cdot |w|^j \geq (1-d)\,|A(e^{i\theta})|$$

(△-ineq)

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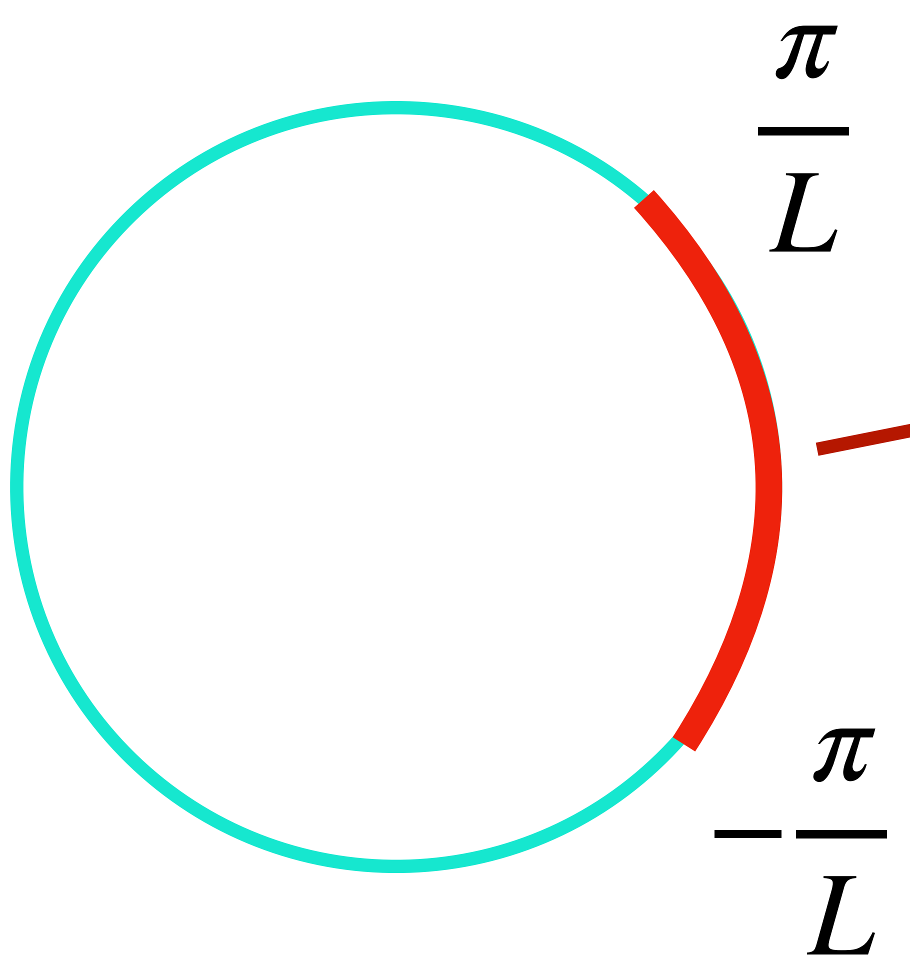

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$$\implies ||\mu(x)||_1 \geq e^{-\frac{c'N}{L^2}} \cdot |A(e^{i\theta})| \qquad \text{(simple trig)}$$

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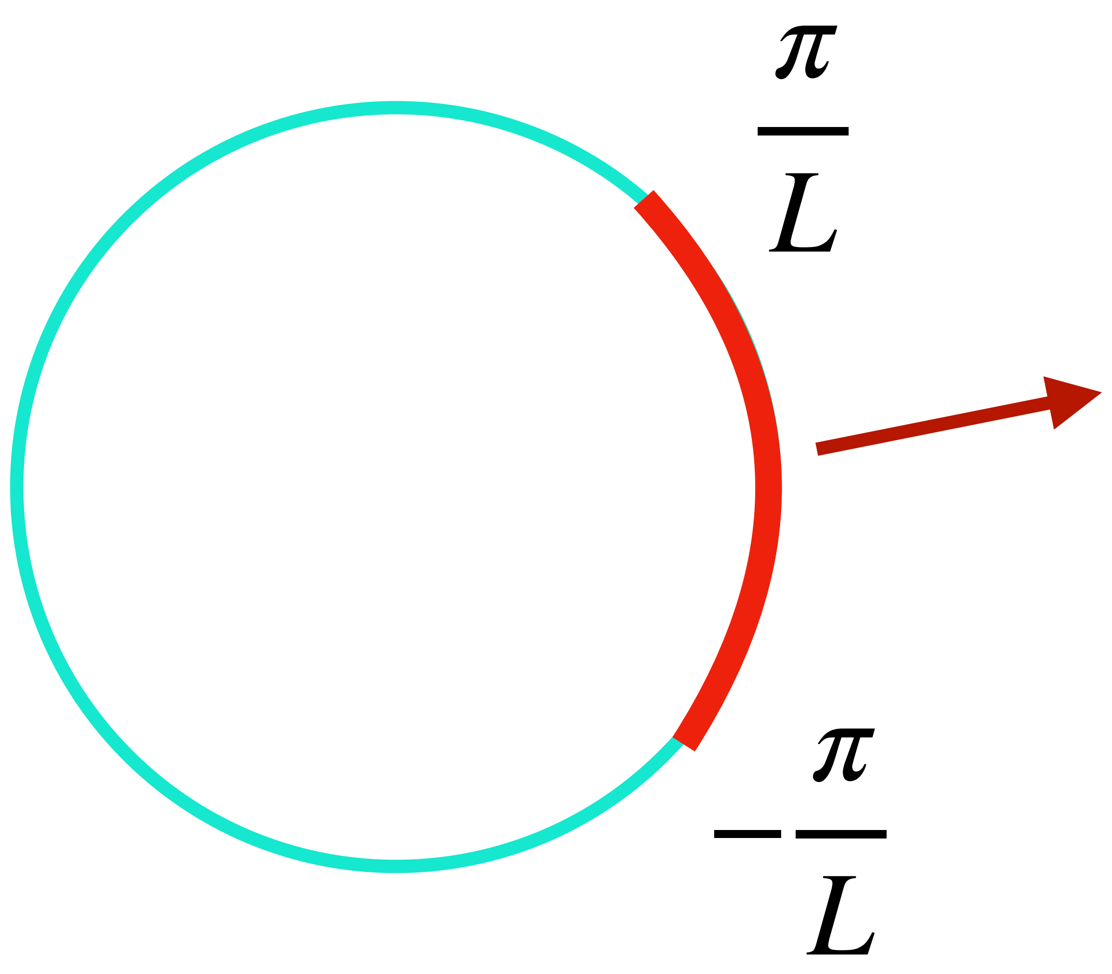

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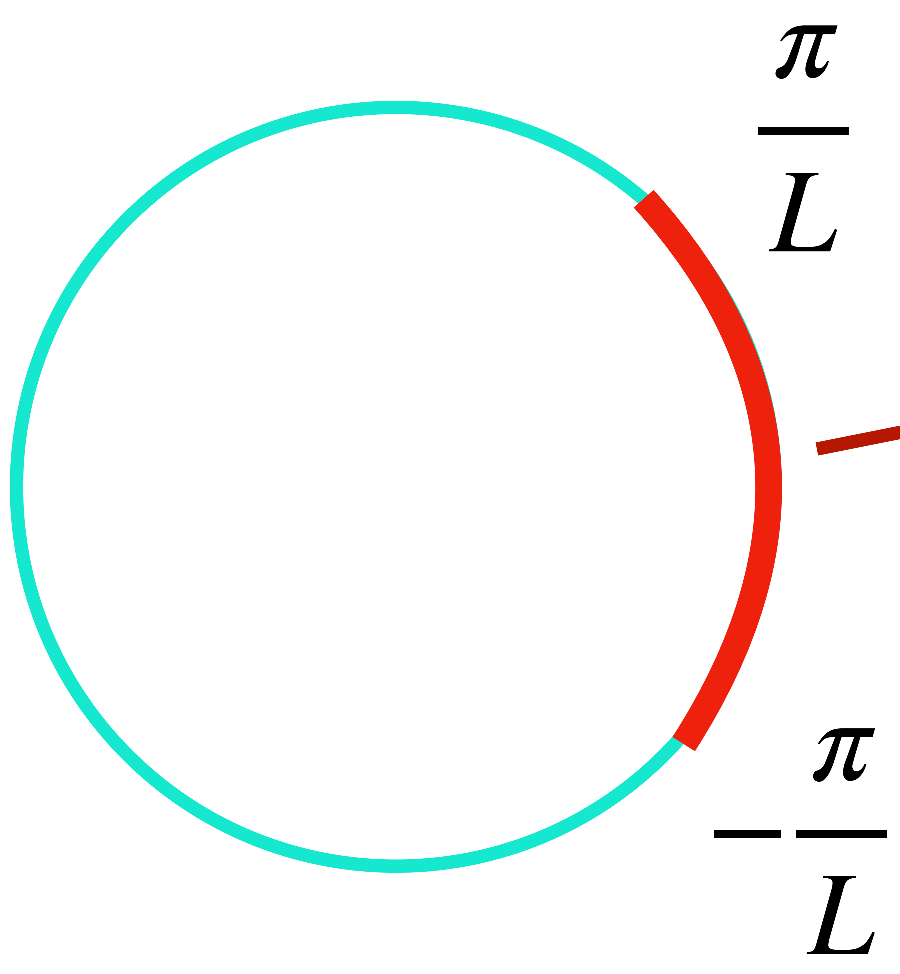

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$$\geq e^{-cN^{1/3}}$$ **([BE97] + Max wrt L)**

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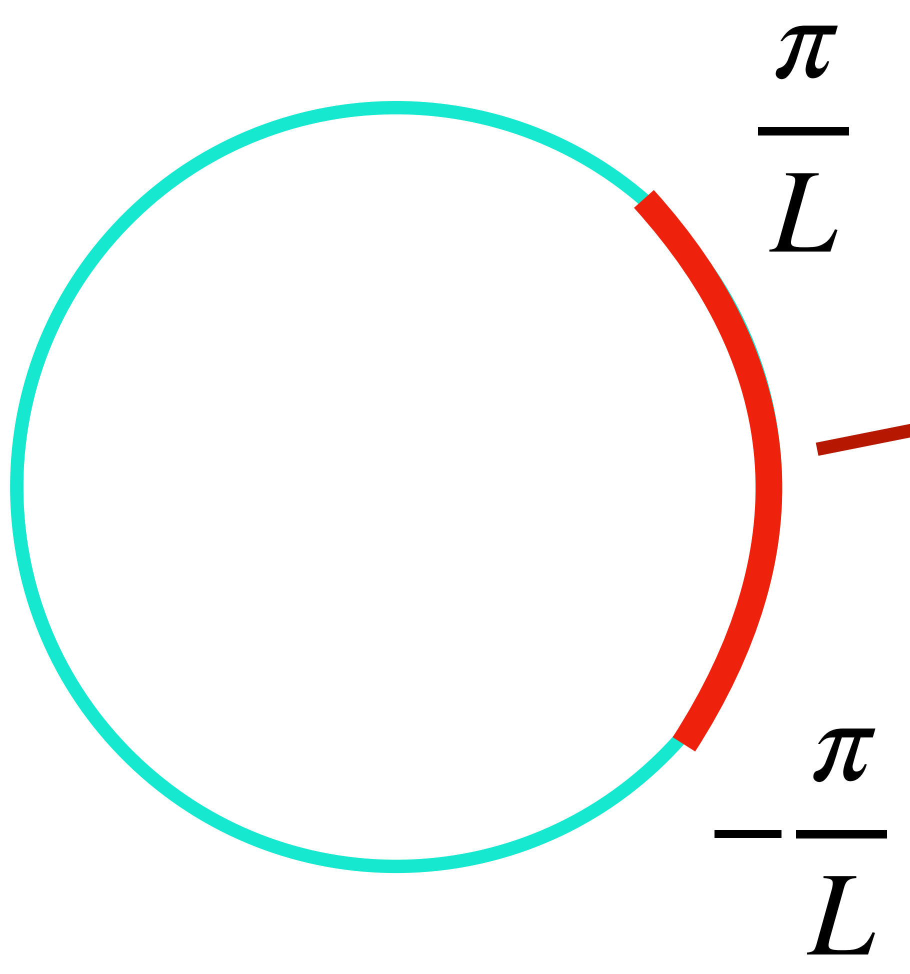

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$$\geq e^{-cN^{1/3}} \qquad \textbf{([BE97] + Max wrt L)}$$

$e^{O(N^{1/3})}$ **traces suffice to distinguish mean traces** $\Longleftarrow$

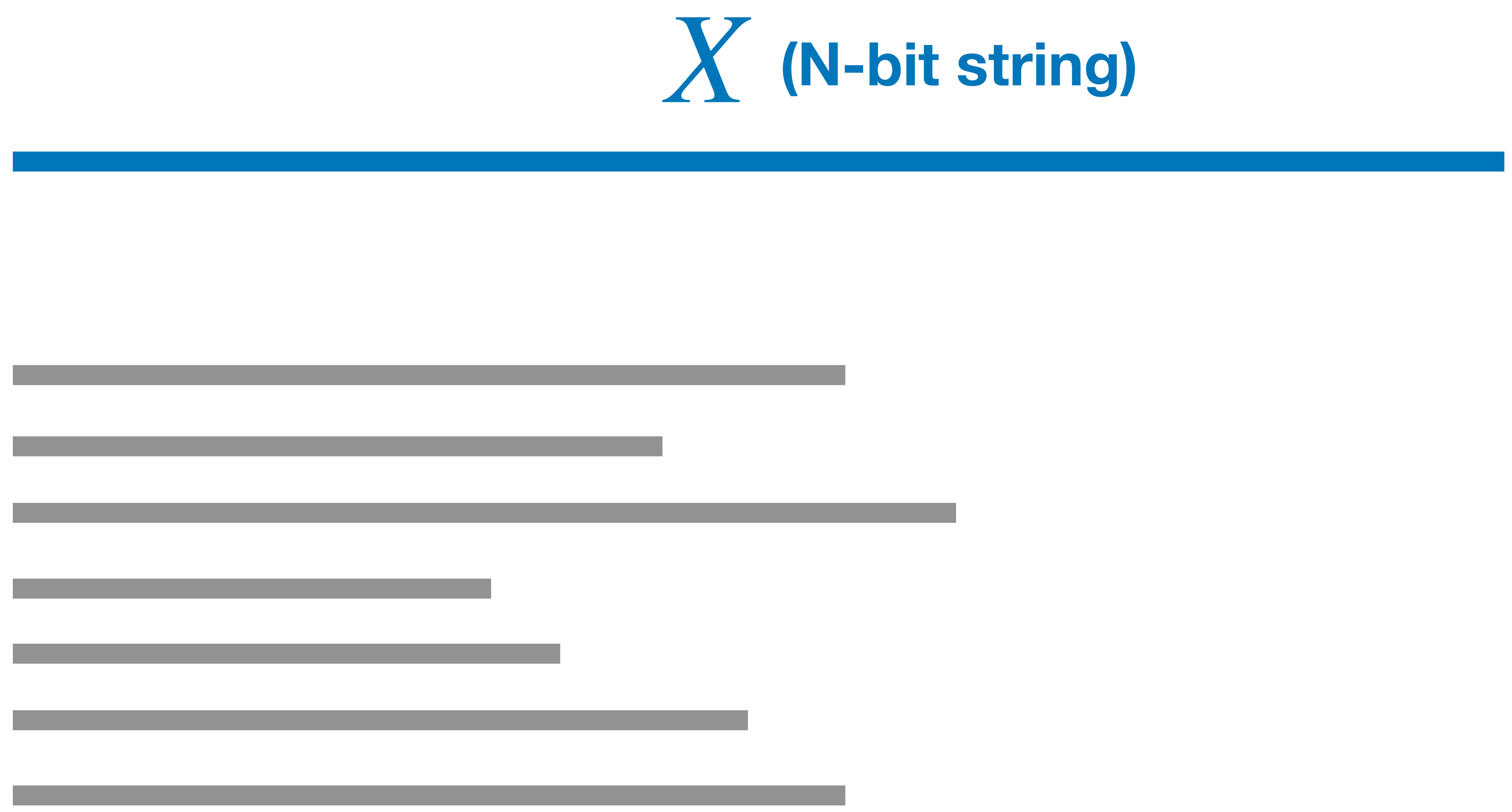
Average-case trace reconstruction

Recall: Average error probability of reconstruction algorithm must be $O(1/N)$.

Reconstruction algorithms	#traces	deletion probability
Batu, Kannan, Khanna, McGregor 2004	$\log N$	$d = 1/\log N$
Holenstein, Mitzenmacher, Panigrahy, Wieder 2008	**poly**(N)	$d < c$ for small absolute constant c
Peres, Zhai 2017	$\exp(\log^{1/2} N)$	$d < 1/2$
Holden, Pemantle, Peres 2018	$\exp(\log^{1/3} N)$	$d < 1$

A recipe for average-case trace reconstruction

[HolensteinMitzenmacherPanigrahyWieder08]



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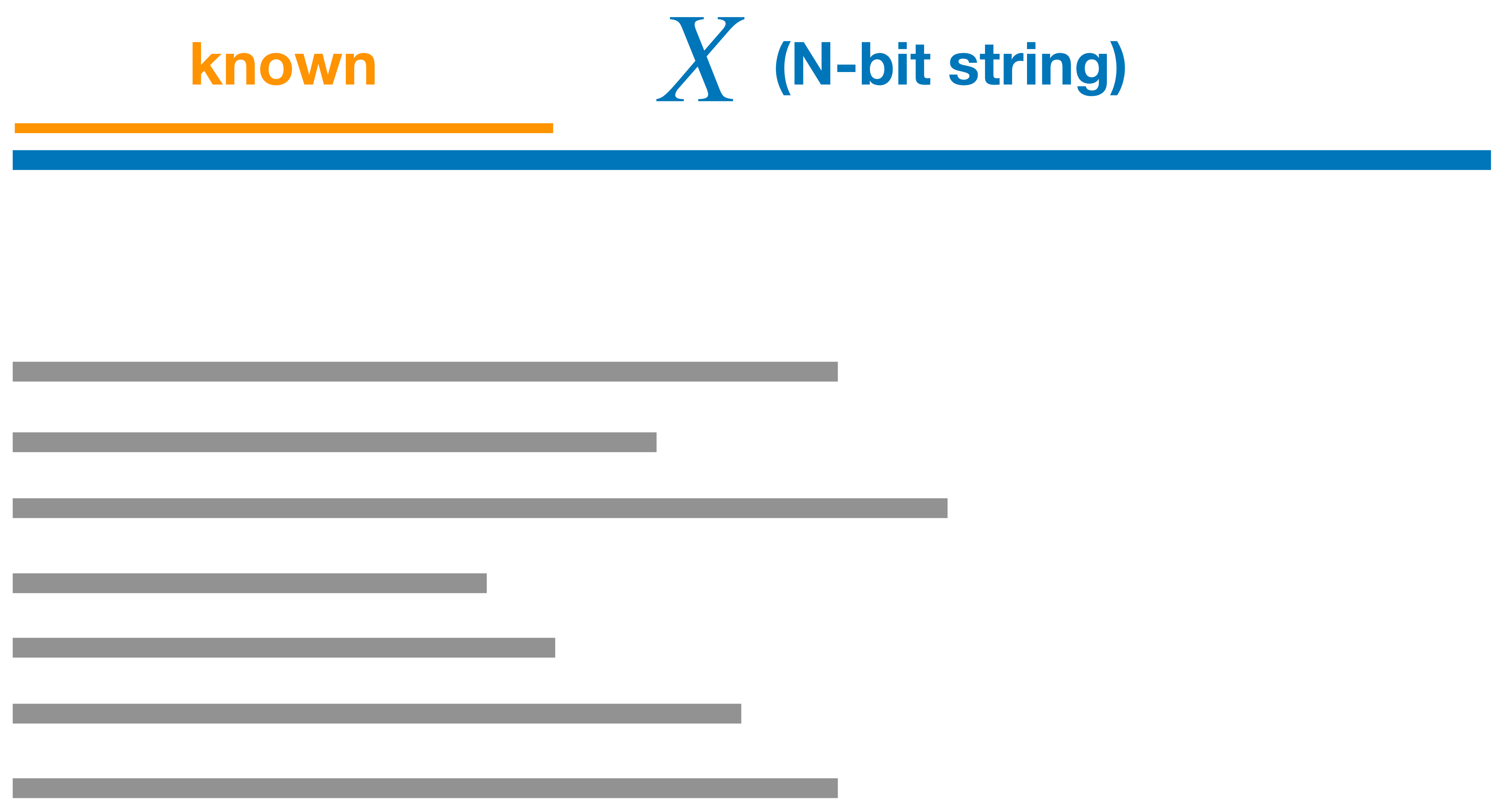
1) Bootstrapping: Learn first bits of X "for free"



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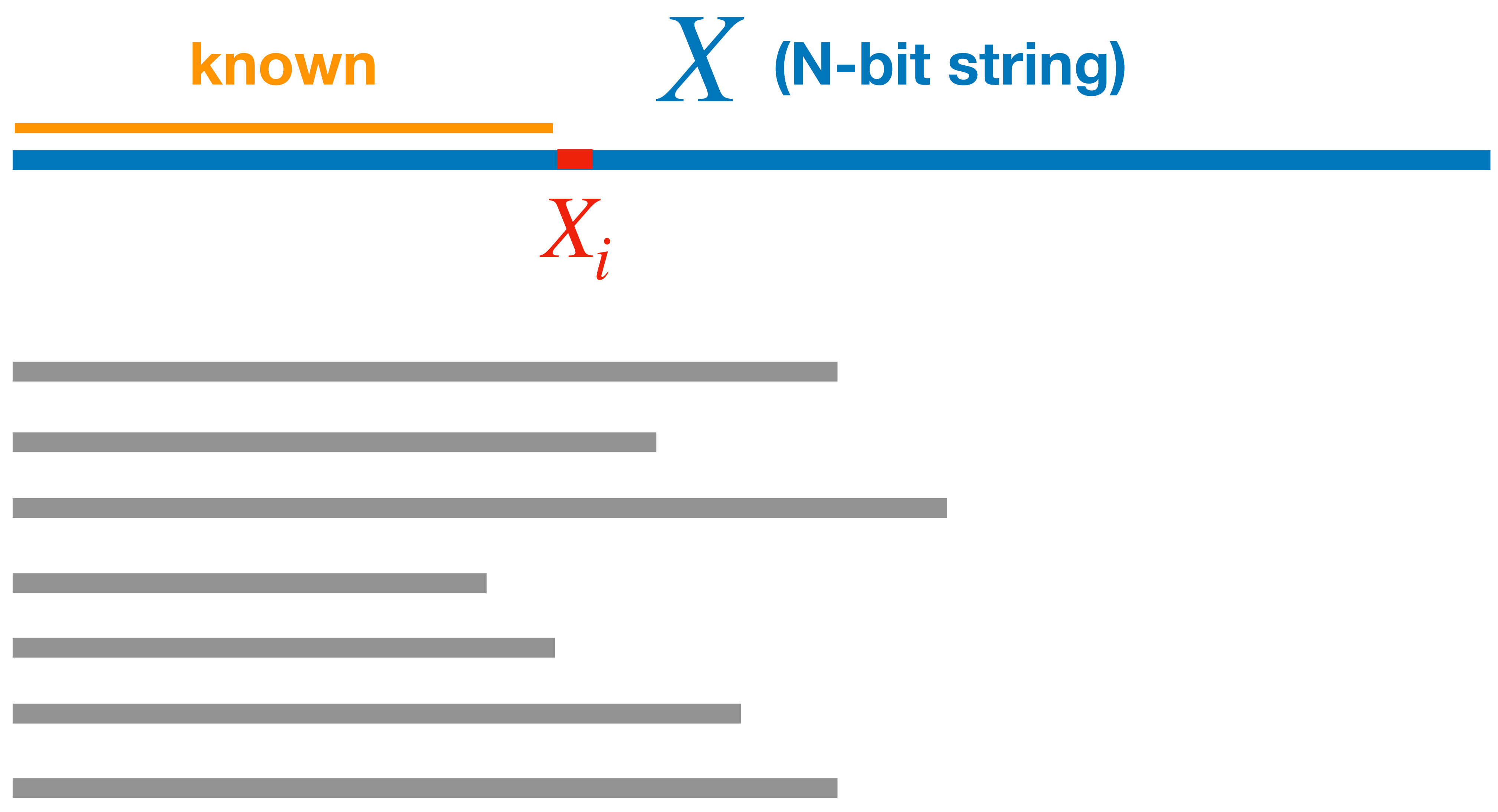
Suppose we know $X_1, X_2, \ldots, X_{i-1}$

Goal: Find X_i



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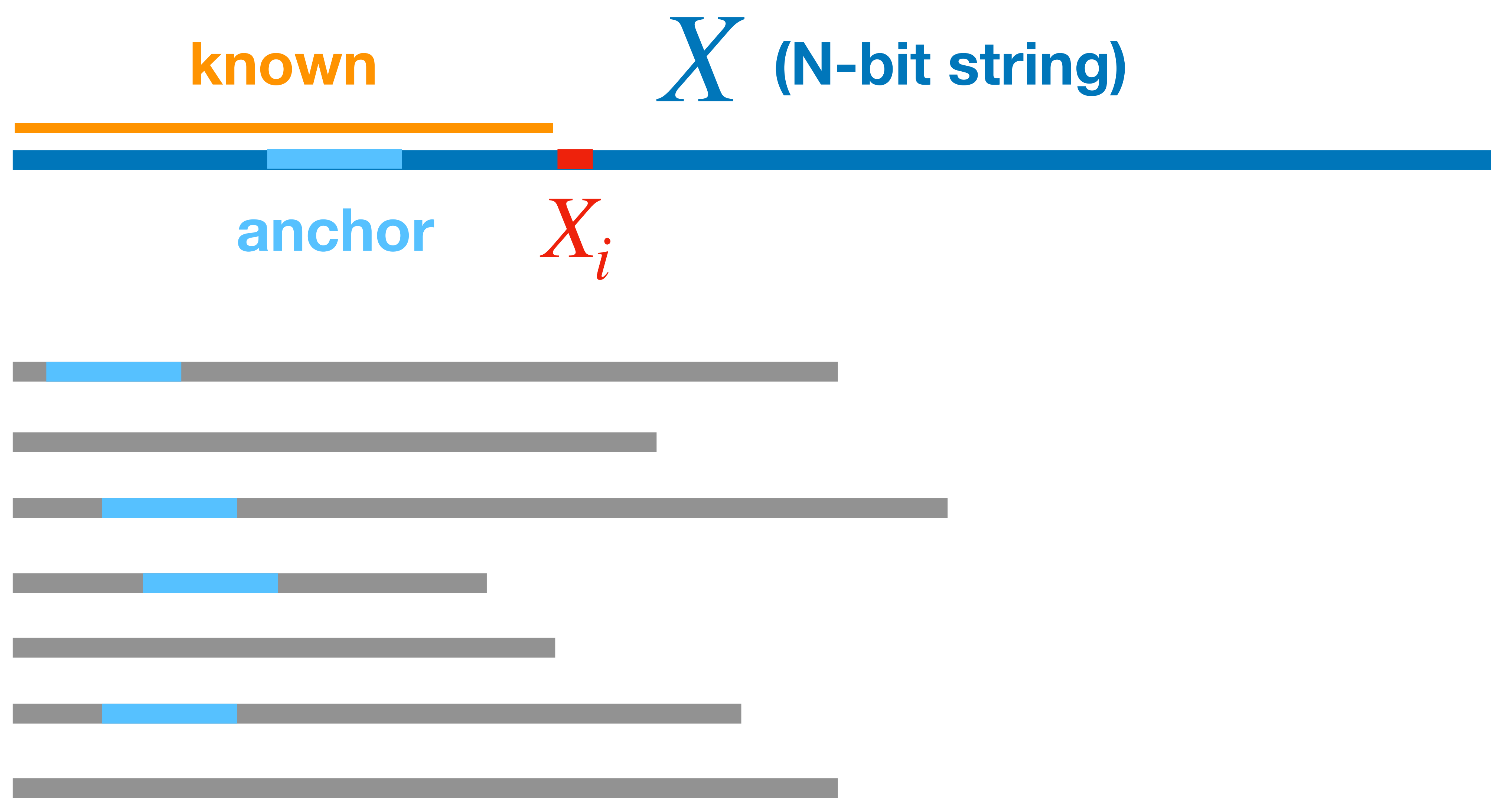
"If X is random, whp anchor in trace comes from anchor in X."



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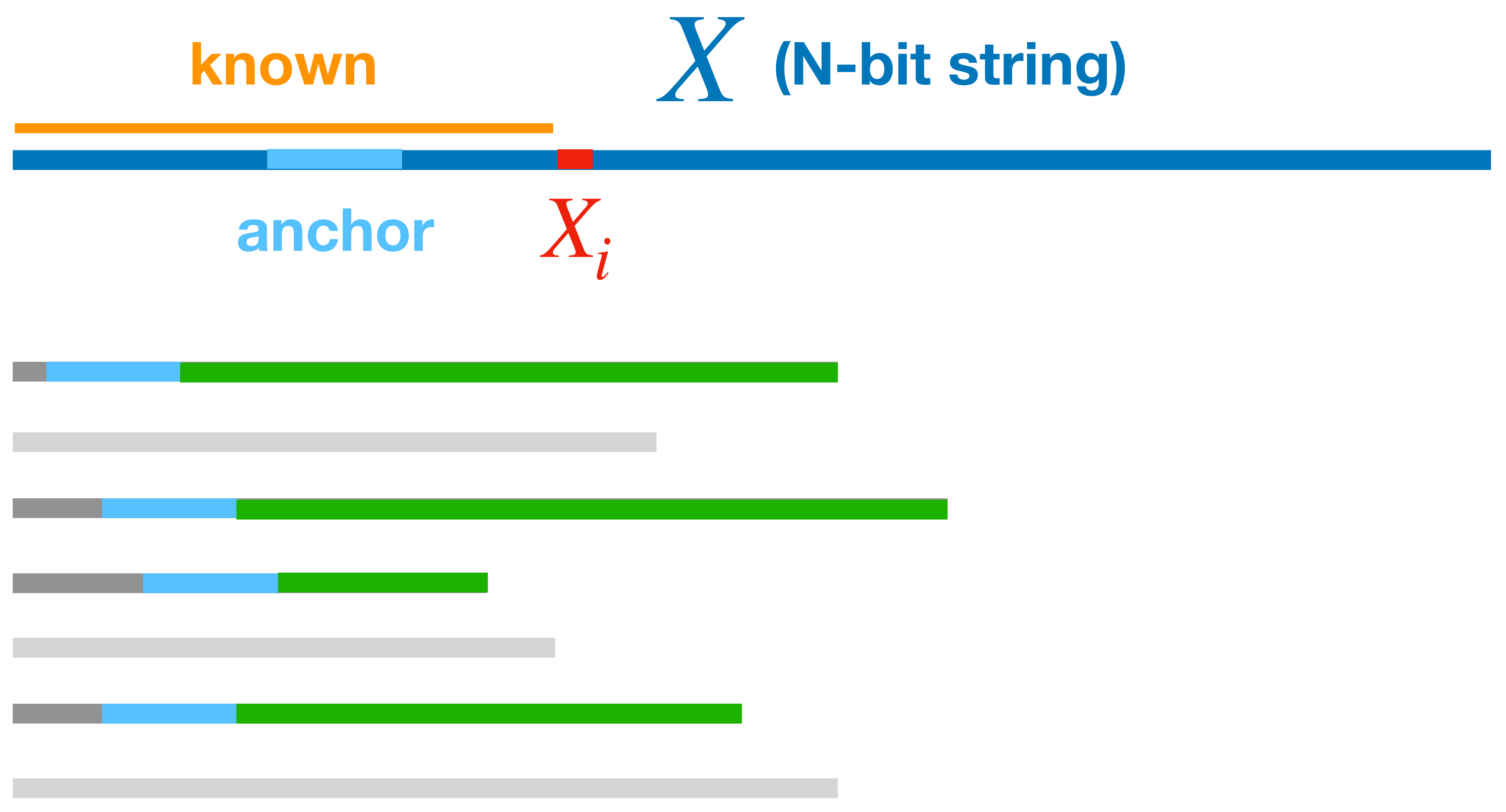

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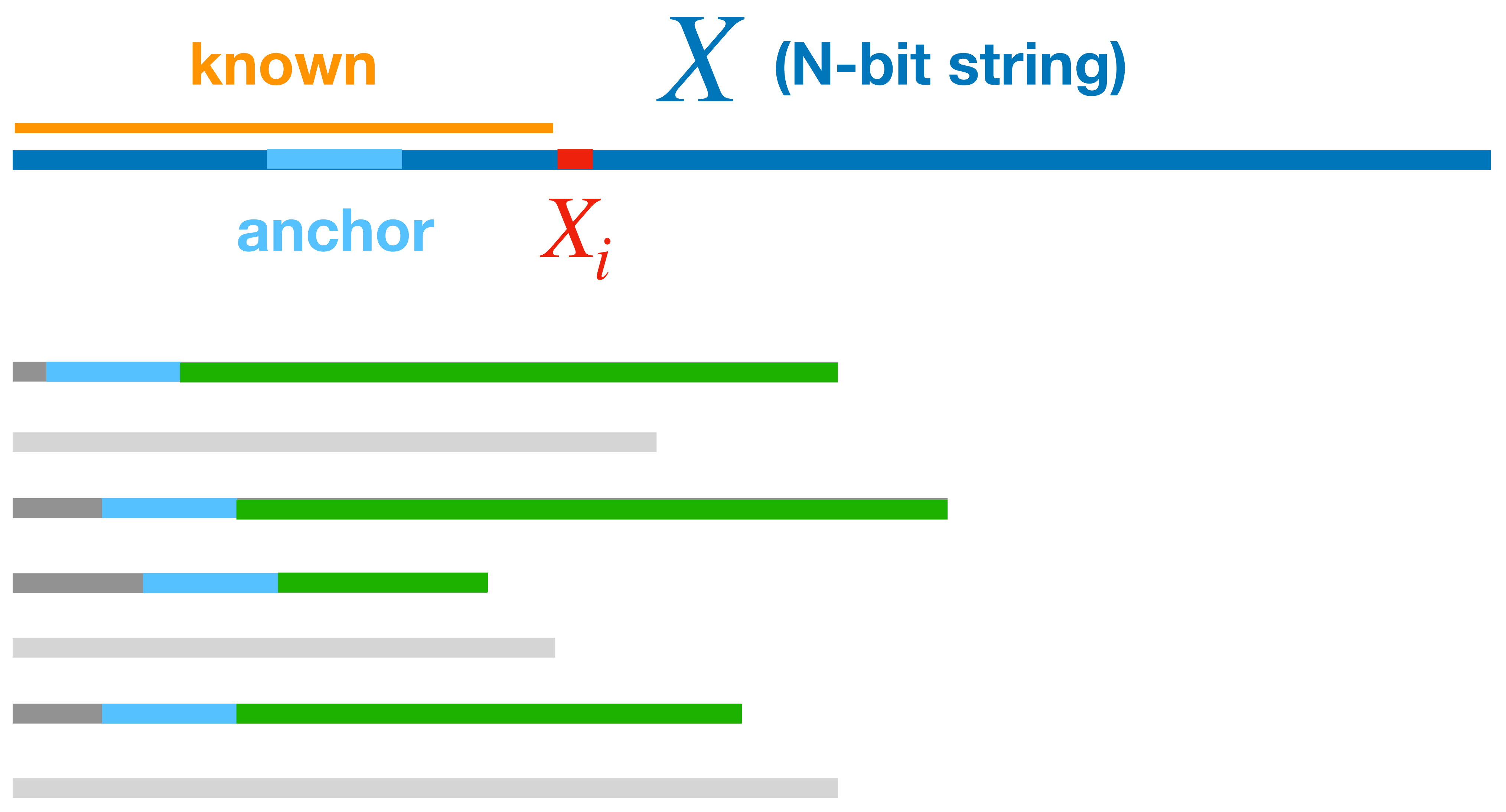

3) Reconstruction: Estimate special bit

$Y =$ distribution of **"trace after anchor"**

"There is special position $Y_{j^\star}$ *that is decently influenced by* X_i *"*

$$|\Pr[Y_{j^\star} = 1 \mid X_i = 1] - \Pr[Y_{j^\star} = 1 \mid X_i = 0]| \geq \frac{1}{N^C}$$

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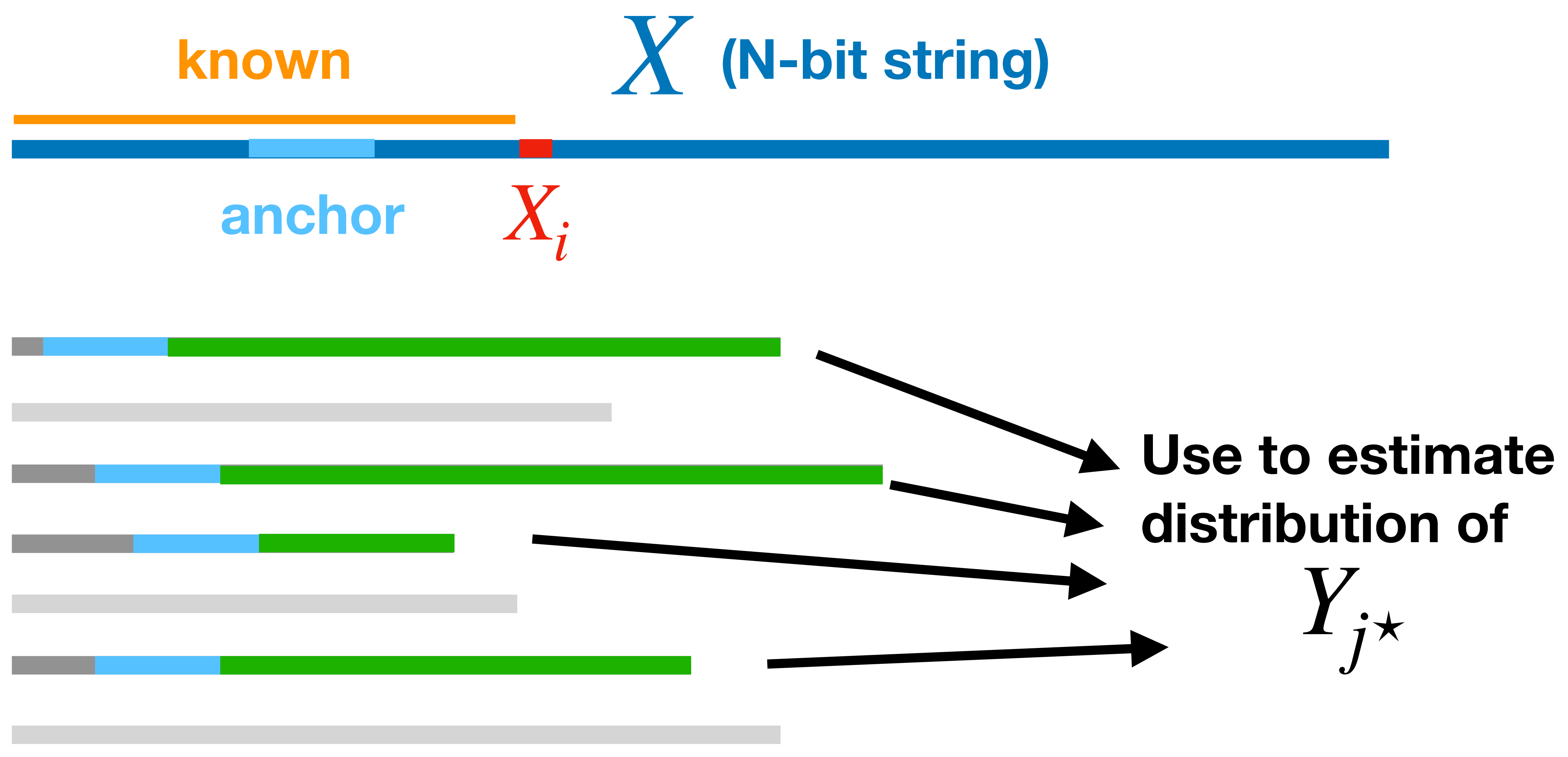

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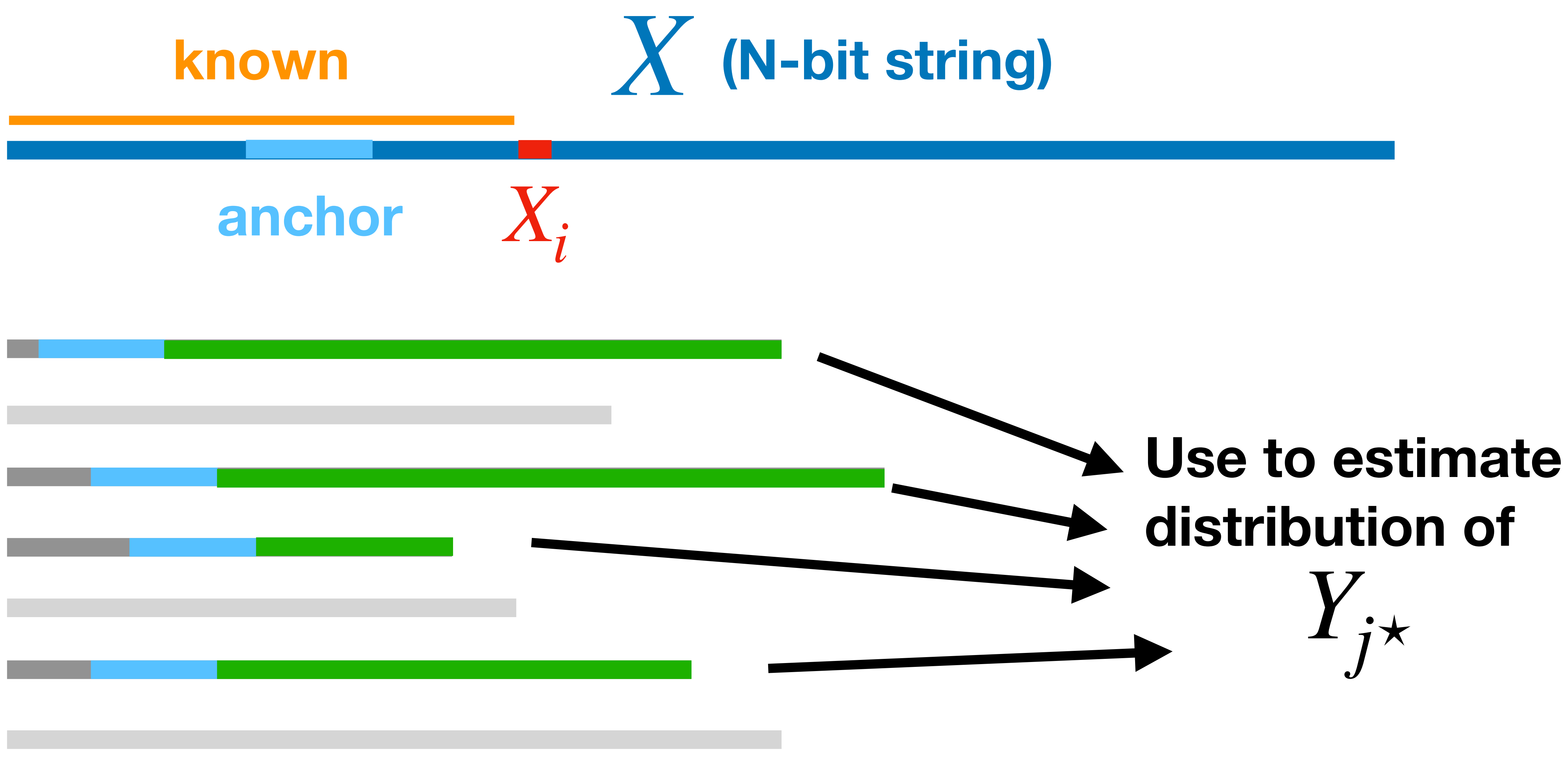

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$$|\Pr[Y_{j^\star} = 1 \mid X_i = \underline{1}] - \Pr[Y_{j^\star} = 1 \mid X_i = \underline{0}]| \geq \frac{1}{N^C}$$

$\boxed{\text{Recover } X_i \text{ whp using } \textbf{poly}(N) \text{ traces!}}$

Lower bounds for trace reconstruction

General recipe for lower bounds:

- **Worst-case:** Show it is hard to distinguish between two specific strings with few traces;
- **Average-case:** Worst-case LB + we expect bad string of length $\frac{\log N}{2}$ to show up $\approx \sqrt{N}$ times in random N-bit string.

Lower bounds #traces	Worst-case	Average-case
McGregor, Price, Vorotnikova 2014	N	$\log^2 N$
Holden, Lyons 2018	$N^{1.25}$	$\log^{2.25} N$
Chase 2019	$N^{1.5}$	$\log^{2.5} N$

Lower bounds for trace reconstruction

[McGregorPriceVorotnikova14]

N traces **necessary and sufficient** to distinguish

$0^{N-1}10^{N}$ **vs.** $0^{N}10^{N-1}$

$$0000\boxed{10}0000$$
$$0000\boxed{01}0000$$

[HoldenLyons18, Chase19]

$N^{1.5}$ traces **necessary and sufficient** to distinguish

$(01)^{N-1}10(01)^{N}$ **vs.** $(01)^{N}10(01)^{N-1}$

$$0101\boxed{1001}0101$$
$$0101\boxed{0110}0101$$

Summing up…

	Worst-case trace reconstruction	Average-case trace reconstruction
Upper bounds #traces	$\exp(O(N^{1/3}))$ [DeO'DonnellServedio17, NazarovPeres17]	$\exp(O(\log^{1/3} N))$ [HoldenPemantlePeres18]
Lower bounds #traces	$\approx N^{3/2}$ [HoldenLyons18, Chase19]	$\approx \log^{5/2} N$ [HoldenLyons18, Chase19]

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almost exponential gap

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	Worst-case trace reconstruction	Average-case trace reconstruction
Upper bounds #traces	$\exp(O(N^{1/3}))$ [DeO'DonnellServedio17, NazarovPeres17]	$\exp(O(\log^{1/3} N))$ [HoldenPemantlePeres18]
Lower bounds #traces	$\approx N^{3/2}$ [HoldenLyons18, Chase19]	$\approx \log^{5/2} N$ [HoldenLyons18, Chase19]

almost exponential gap

***All results over $\{0,1\}^N$, as it is the hardest setting**

Trace reconstruction and portable DNA-based storage

[YazdiGabrysMilenkovic17, Organick+18]

Write process: Data is encoded into {A,G,T,C} alphabet, then synthesized into DNA strand.

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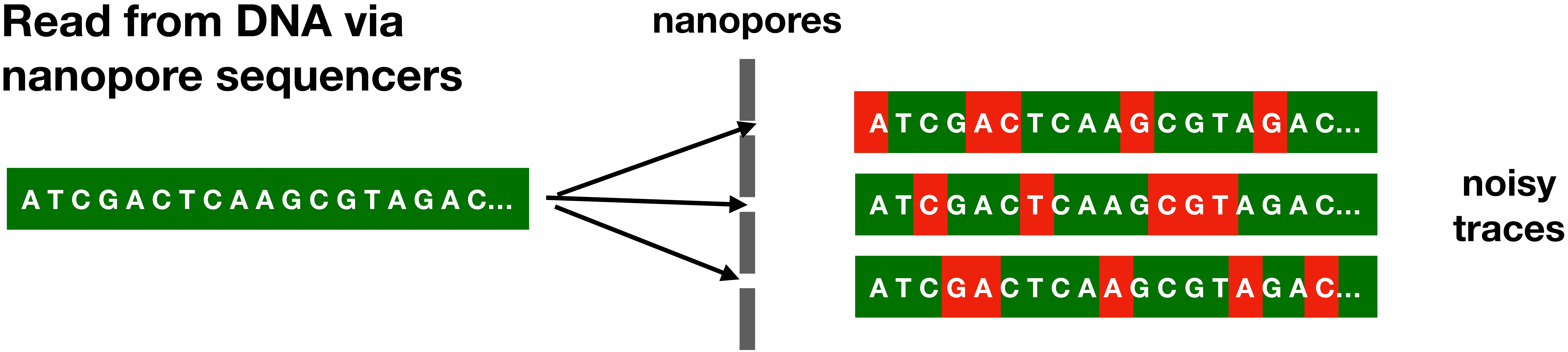
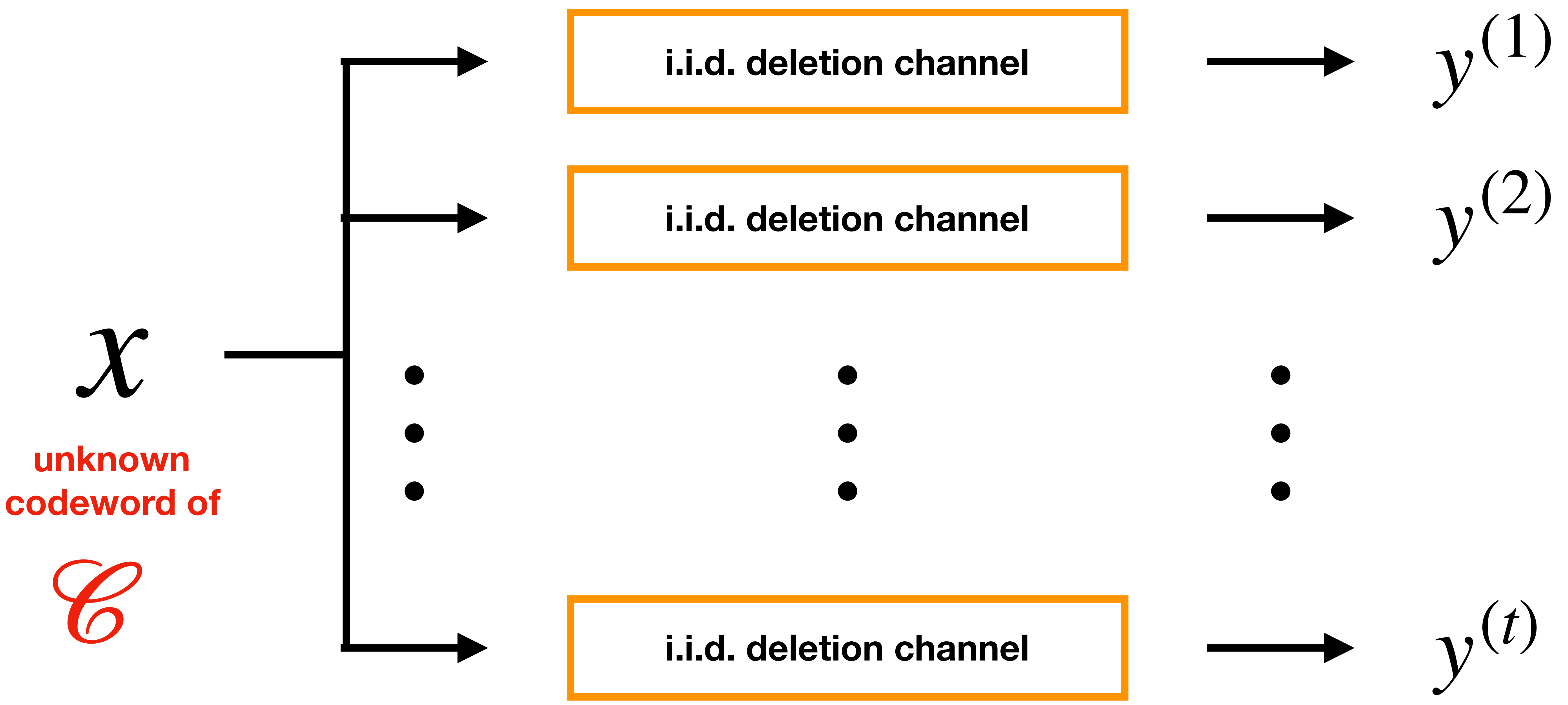
Read from DNA via nanopore sequencers



Trace reconstruction for the coding schemes in [YGM17, OAC+18] is based on heuristics.

Coded trace reconstruction

[CheraghchiGabrysMilenkovicRibeiro19]



$\mathscr{C} \subseteq \{0,1\}^N$ **with** $|\mathscr{C}| = 2^{RN}$

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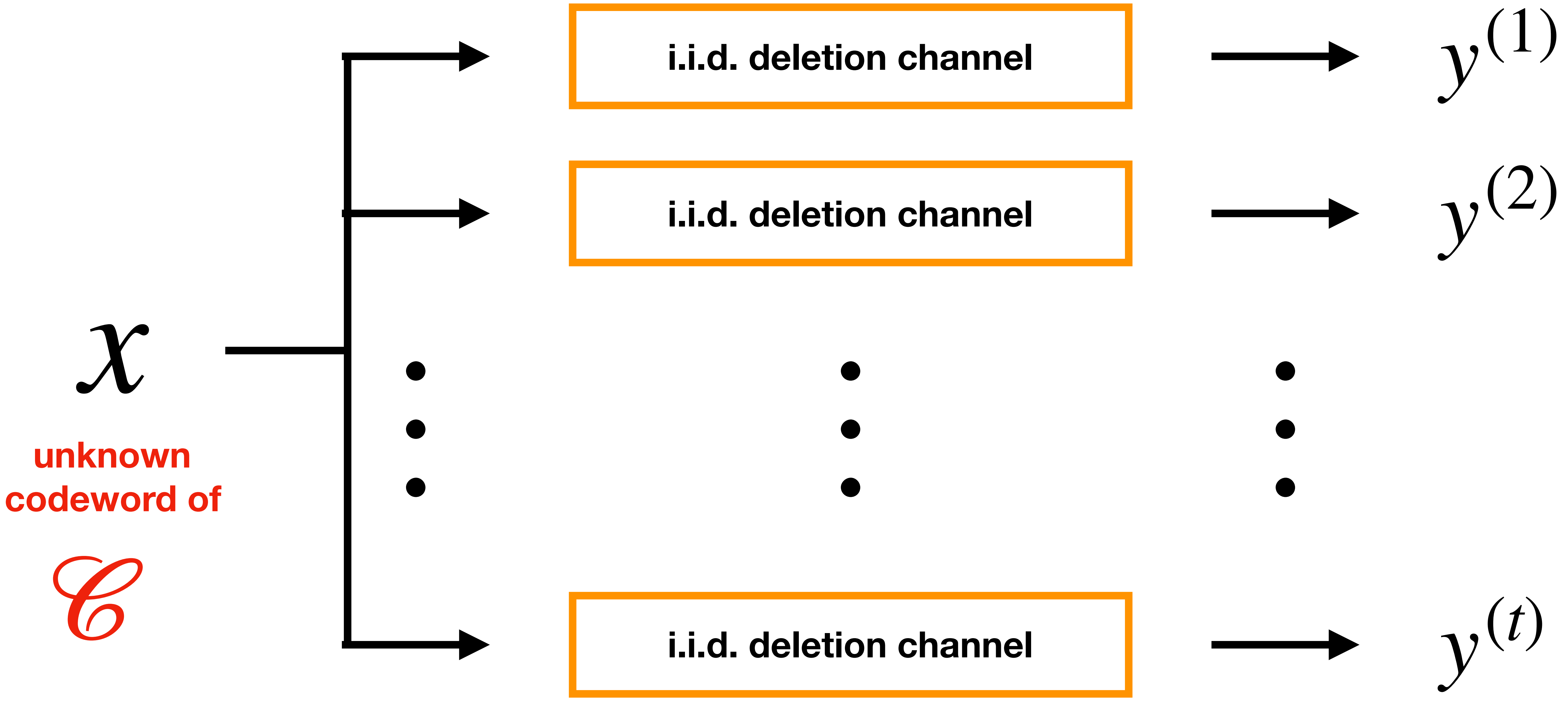

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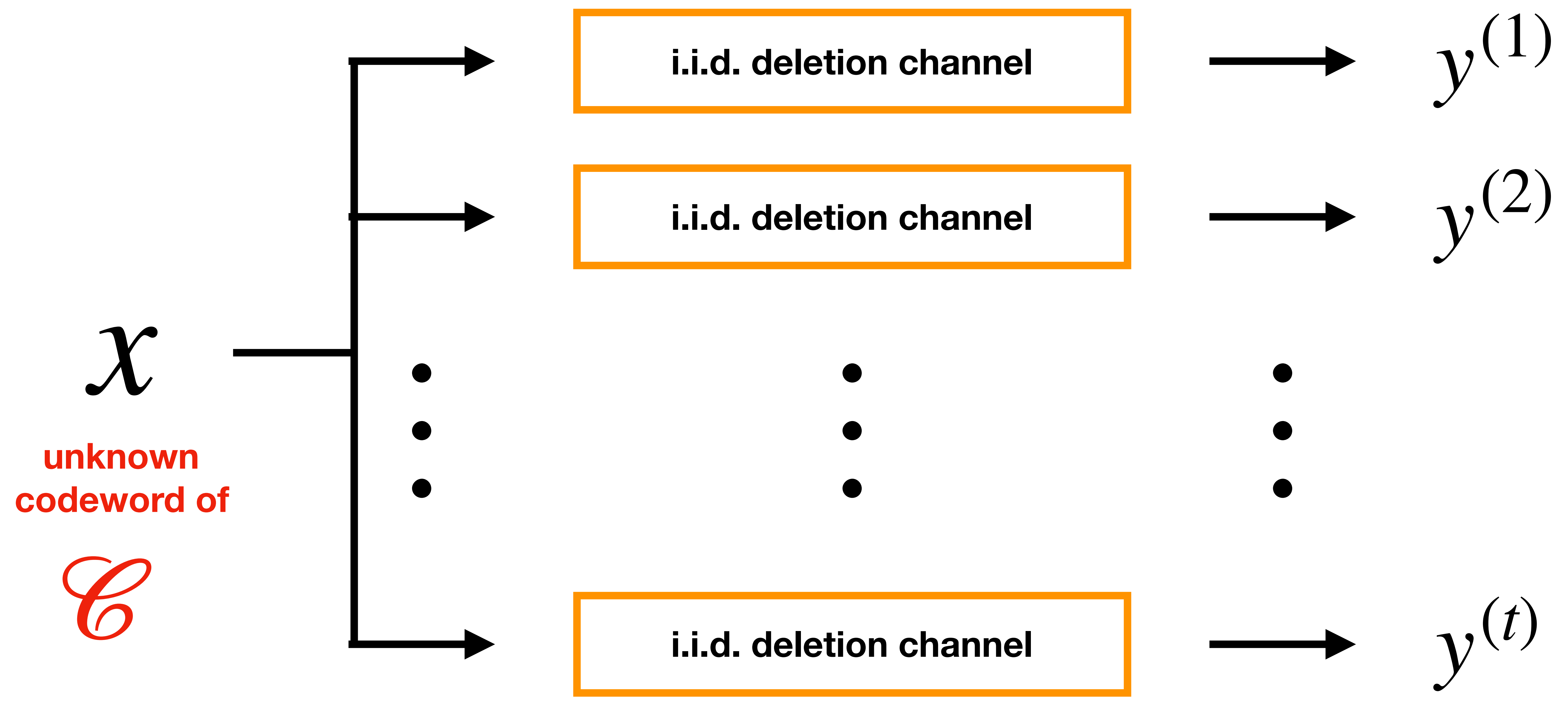

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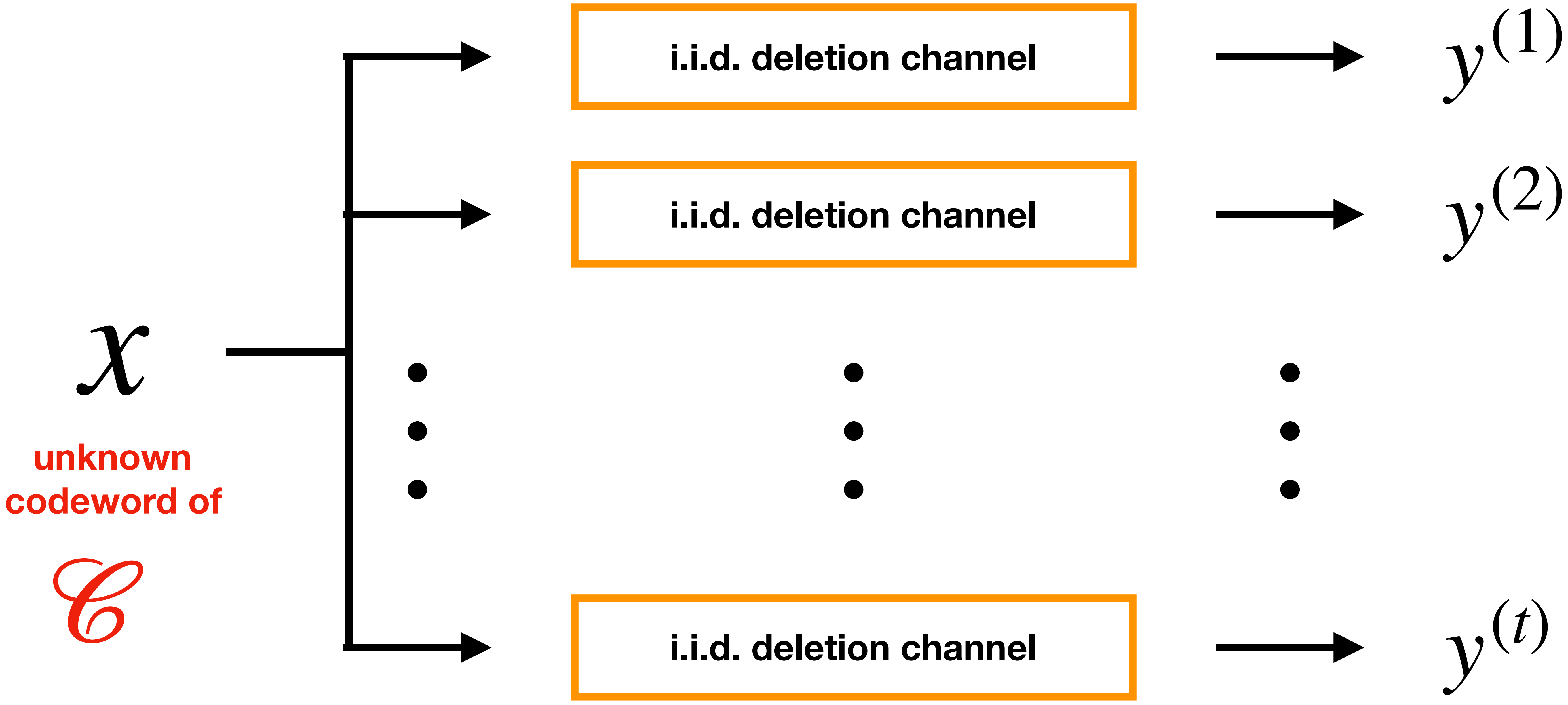

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- high rate $R = 1 - \epsilon, \quad \epsilon = o_N(1)$
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- **efficiently** encodable and reconstructable (time **poly**(N))

Some related work

Haeupler, Mitzenmacher 2014	First-order capacity for small deletion probability **(constant number of traces only)**
Abroshan, Venkataramanan, Dolecek, Guillén i Fàbregas 2019	Coding for multiple deletion channels with **constant number of deletions only**. Uses VT codes. Builds up on work about file synchronization.
Average-case trace reconstruction	Implies existence of large codes reconstructable from few traces. **No efficient encoding + we want to use even fewer traces.**

PART I:

Markers + worst-case trace reconstruction

Basic construction

x N**-bit string**

Basic construction

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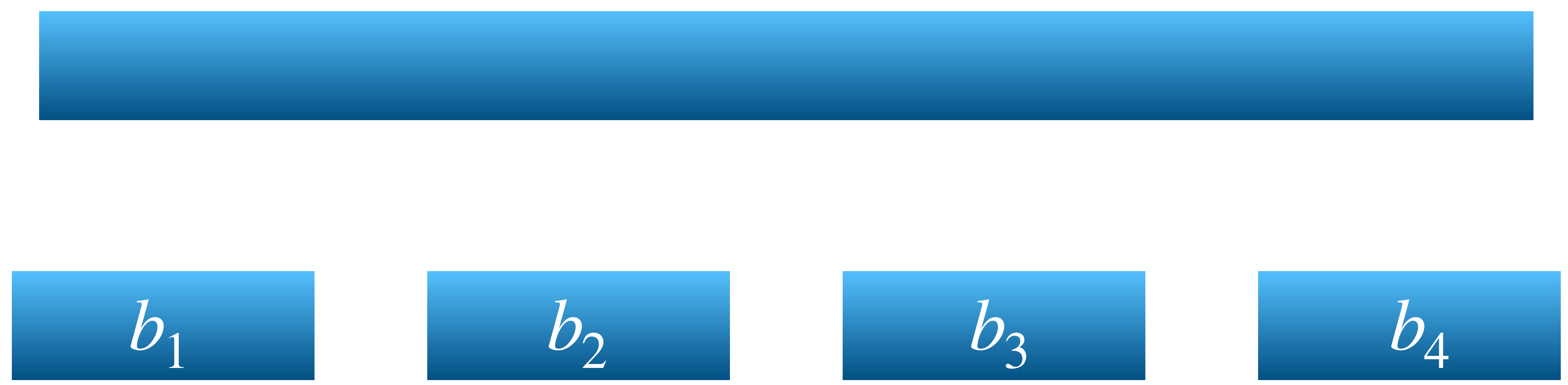
split into blocks with $m = O(\log^2 N)$ **bits**



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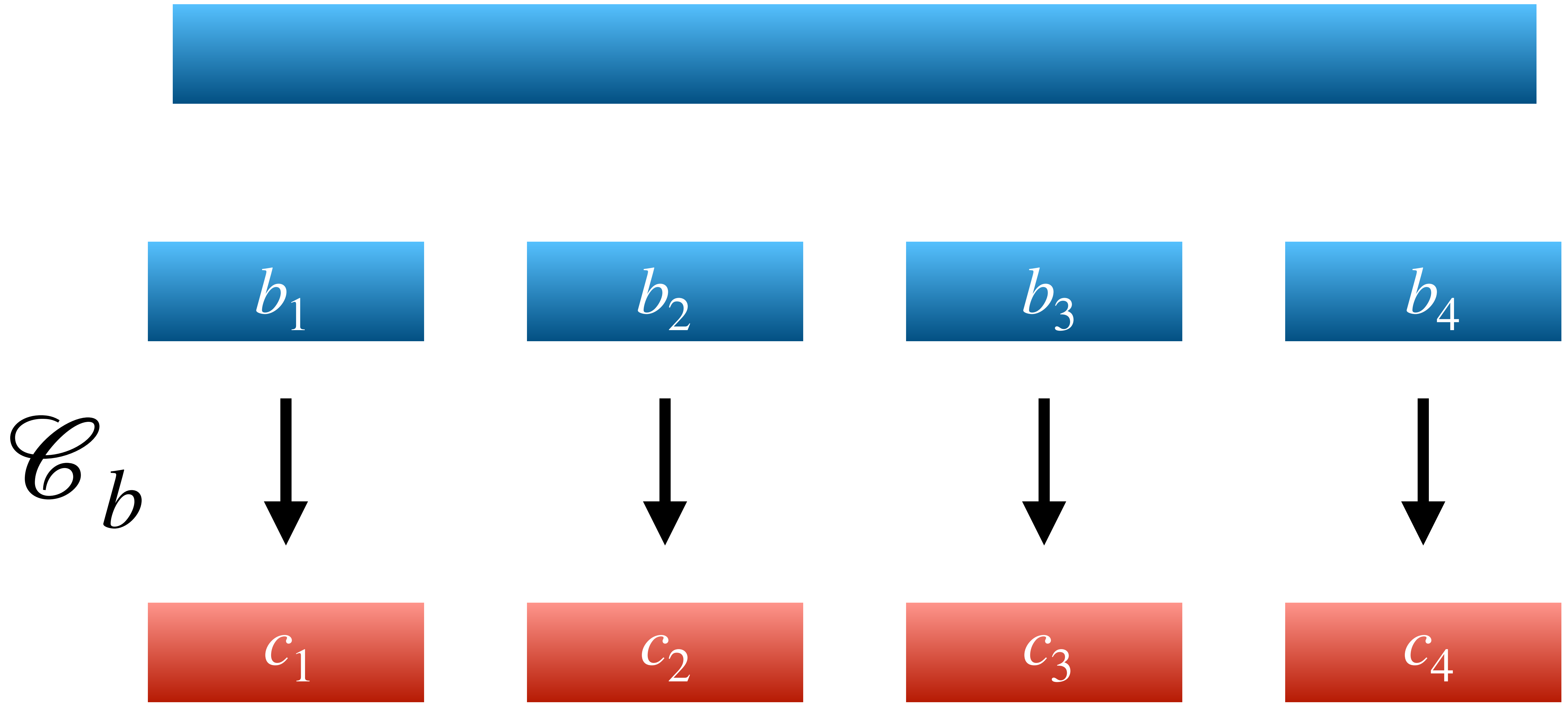

Basic construction

x $\quad$ N**-bit string**

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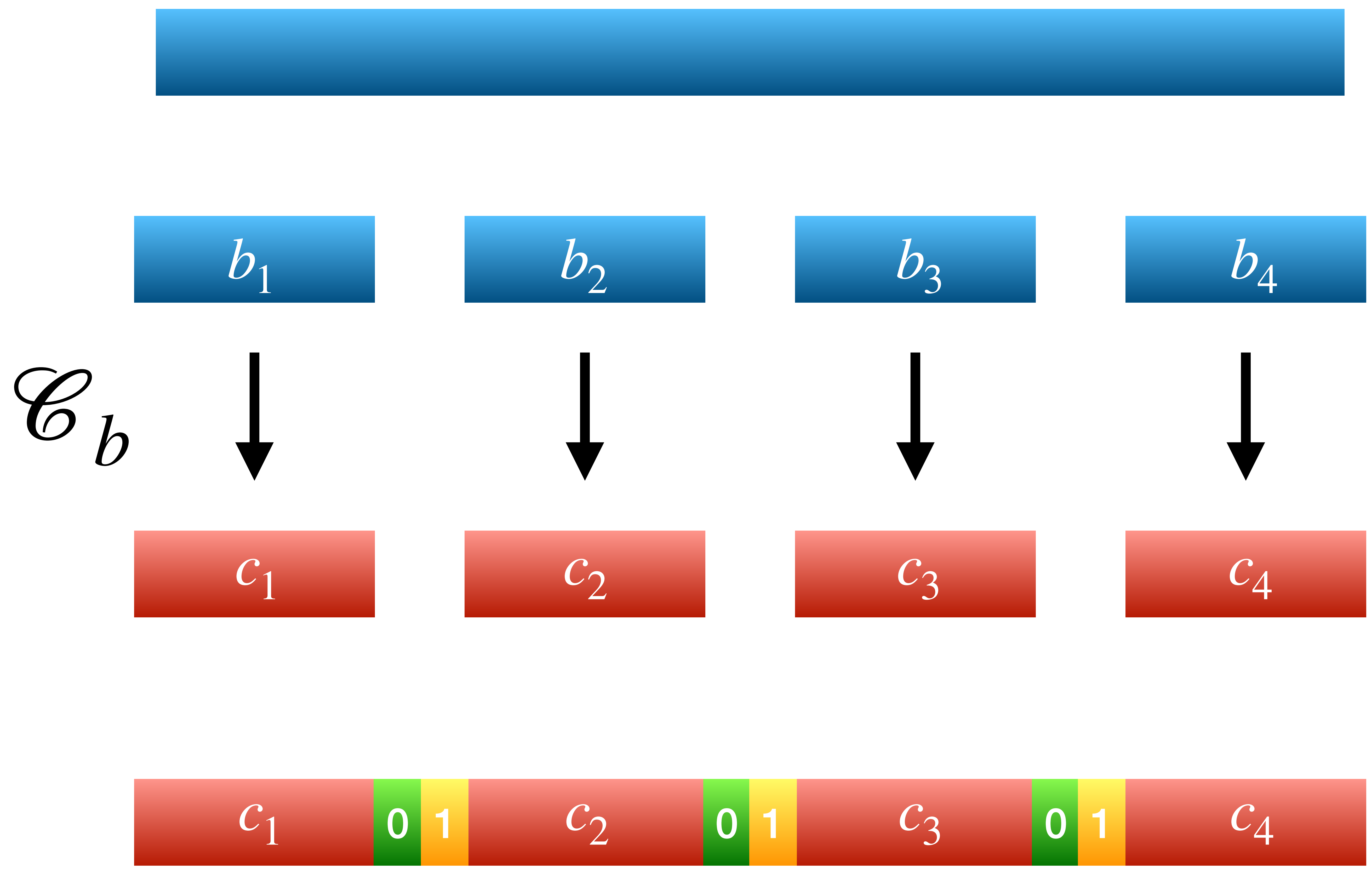
add markers between blocks

0 $= 0000\ldots00$
O(log N) bits

1 $= 1111\ldots11$
O(log N) bits



Reconstruction

$= 0000\ldots00$
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$= 1111\ldots11$
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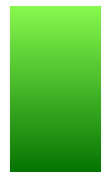
$\in \mathscr{C}_b$

Reconstruction

 $= 0000\ldots00$
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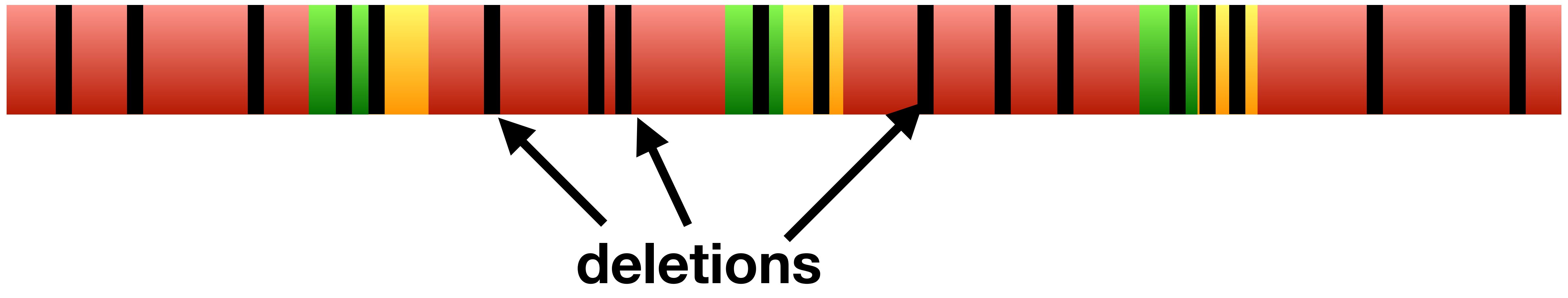

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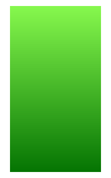 $= 0000\ldots00$

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Trace of codeword

Key observation: part of trace coming from a marker still looks like a marker.

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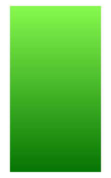 $= 0000\ldots00$

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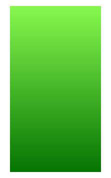 $= 0000\ldots00$
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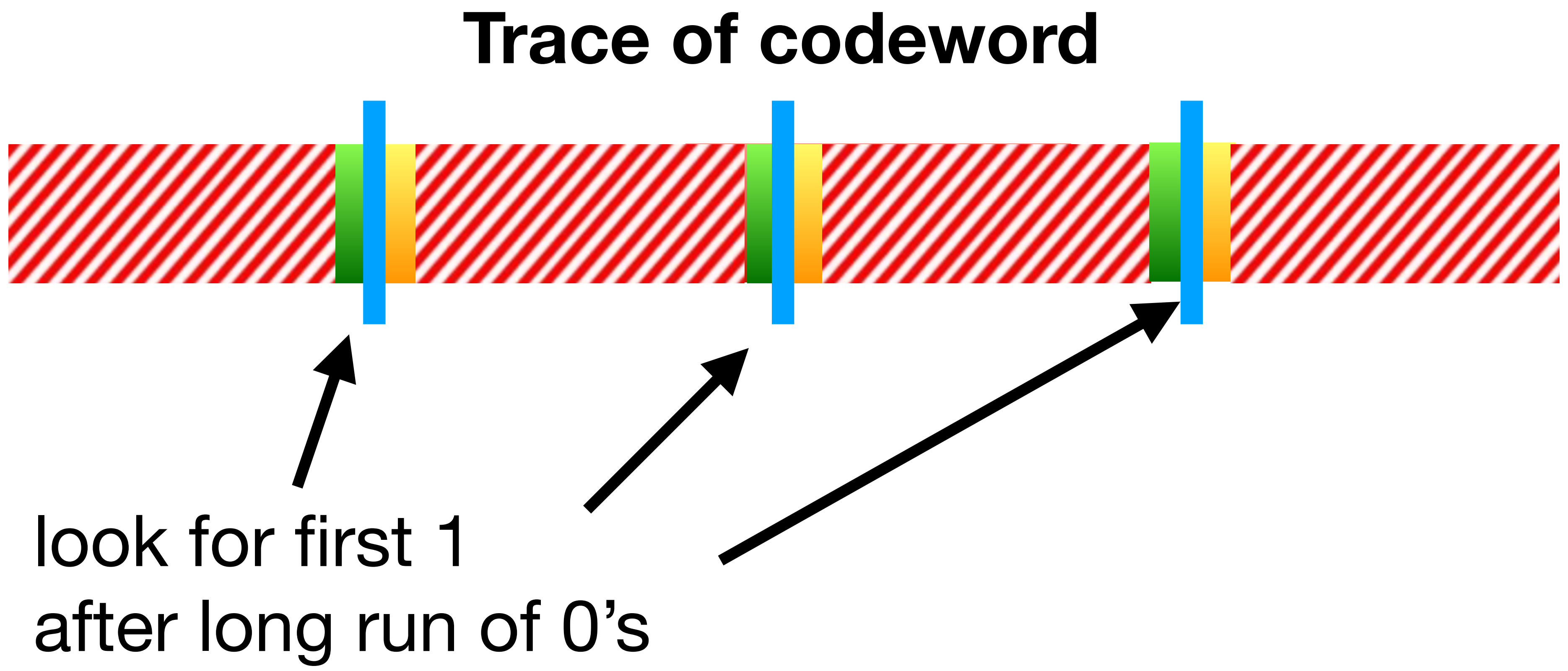

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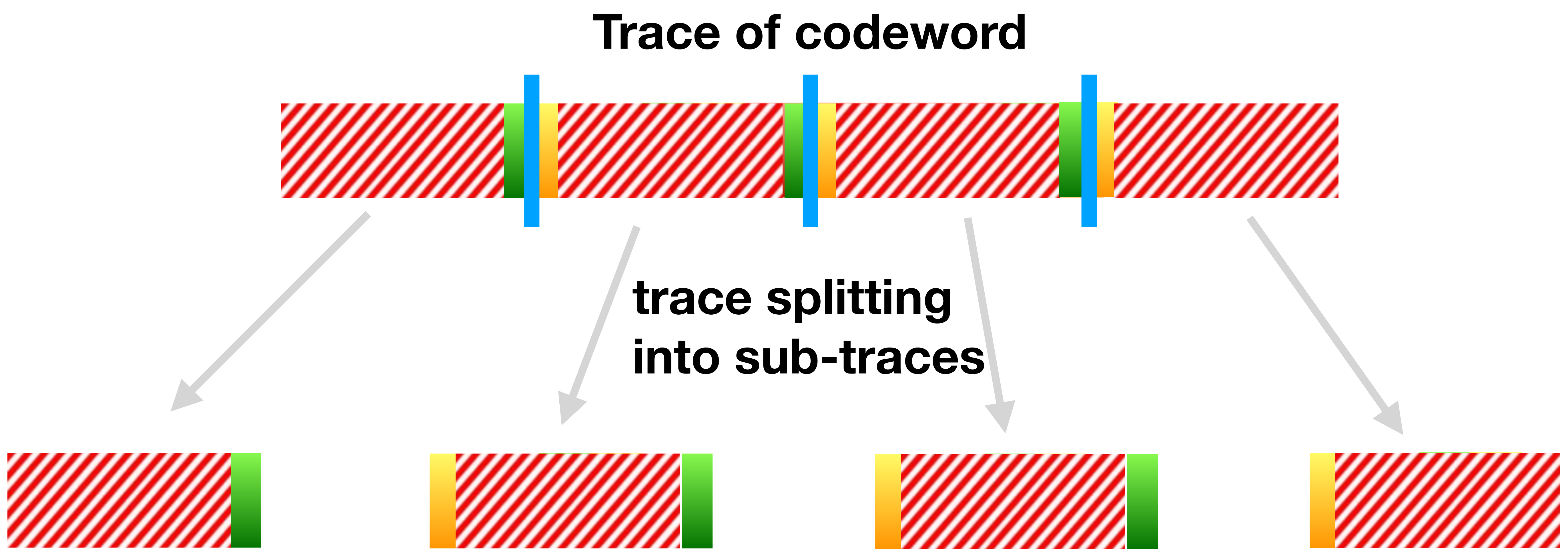

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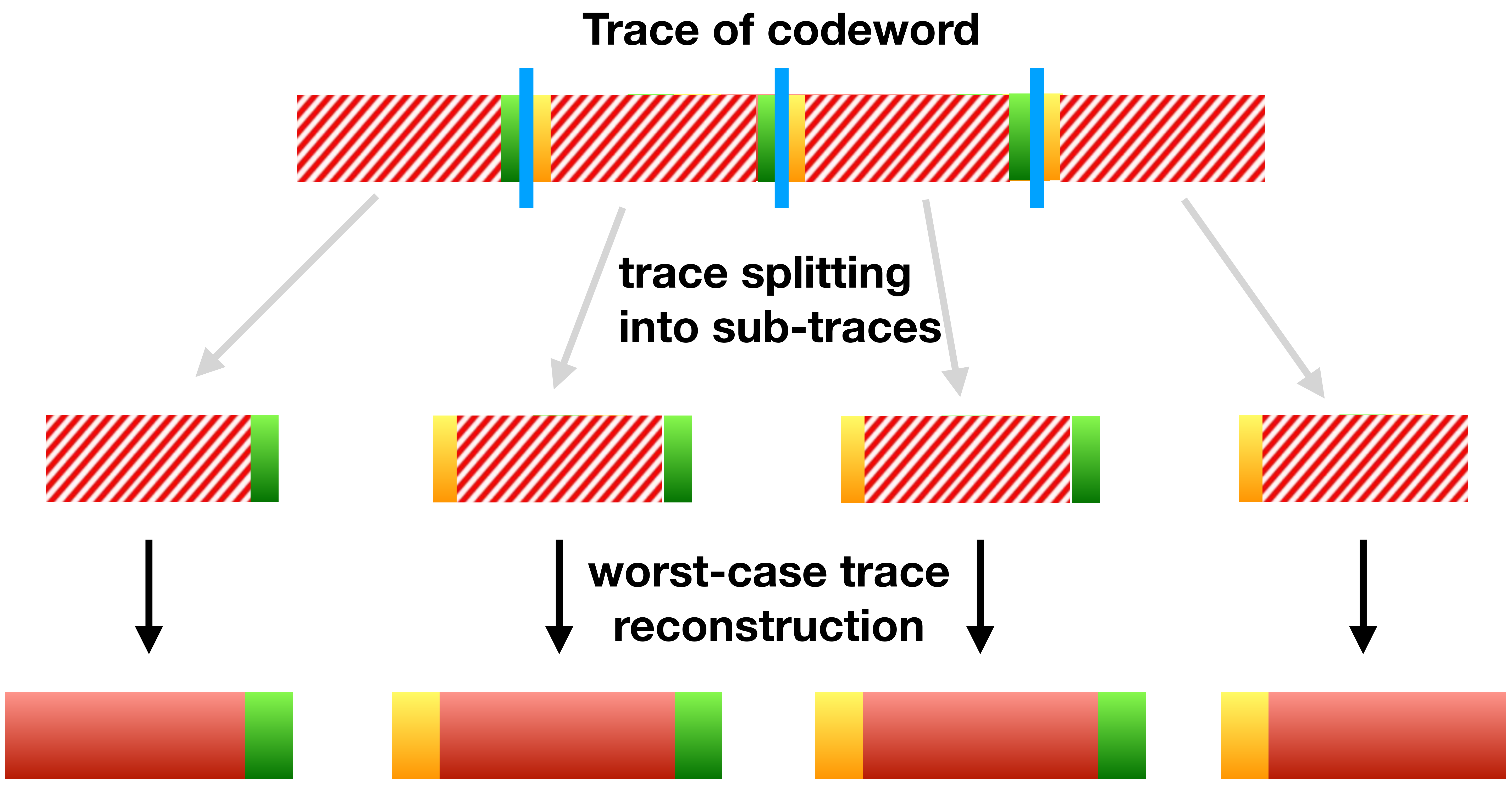

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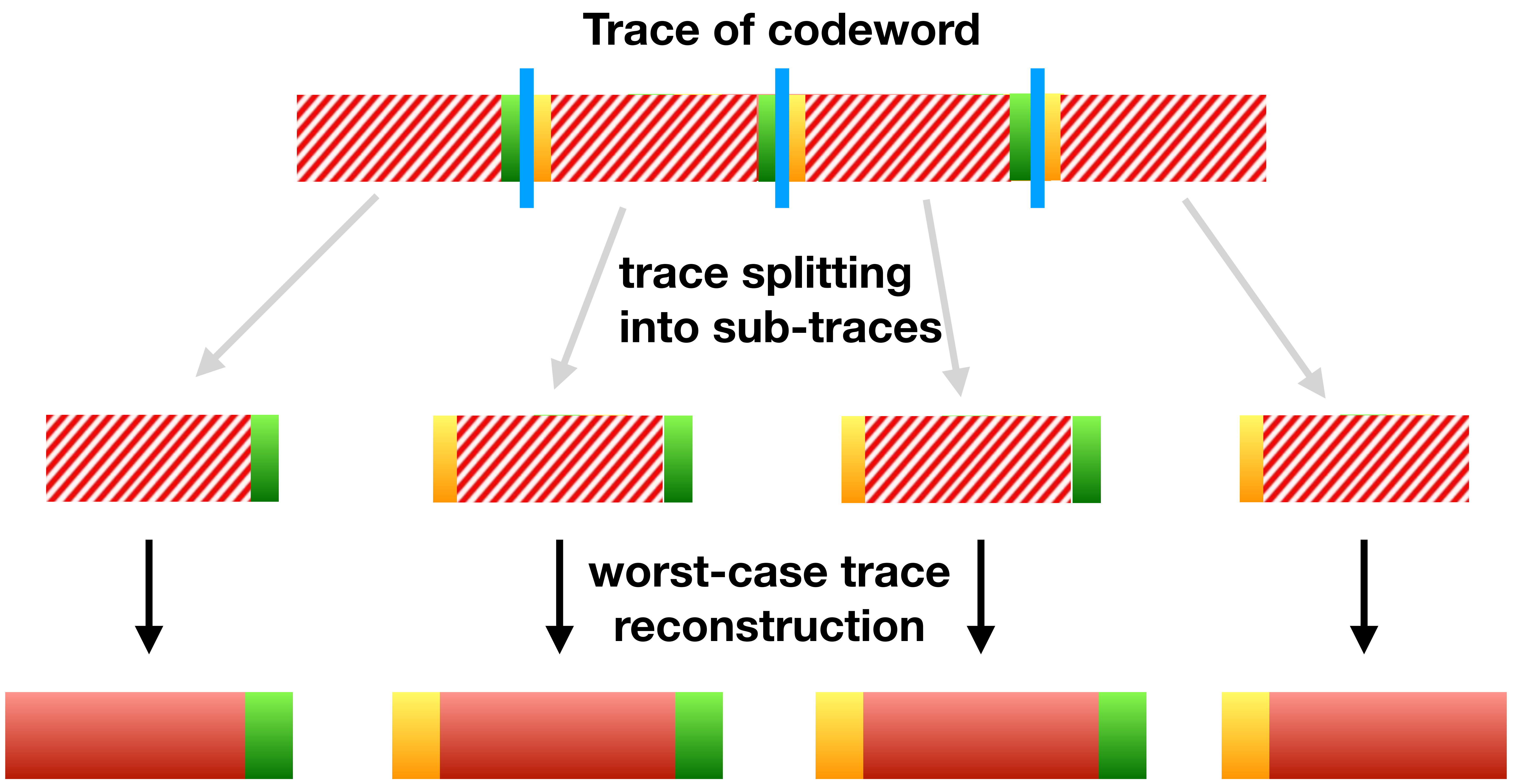

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Efficiently encodable/reconstructable code with rate $1 - O(1/\log N)$ using $\exp(\log^{1/3+\gamma} N)$ traces for any constant deletion prob.

A tool for designing the sub-code: Almost k-wise independent strings

ϵ-almost k-wise independent distribution: $X \in \{0,1\}^m$

$$\forall i_1, \ldots, i_k, x_1, \ldots, x_k : |\Pr[X_{i_1} = x_1, \ldots, X_{i_k} = x_k] - 2^{-k}| \leq \epsilon$$

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For every m, $k = O(\log m)$ and $\epsilon = \frac{1}{\textbf{poly}(m)}$ there is a **poly**(m)-computable function

$$g : \{0,1\}^t \to \{0,1\}^m$$

with $t = O(\log m)$ such that $g(U_t)$ is ϵ-almost k-wise independent.

Designing the sub-code

Desired property: No long runs of 0's in trace of $c \in \mathscr{C}_b$ → allows for trace splitting

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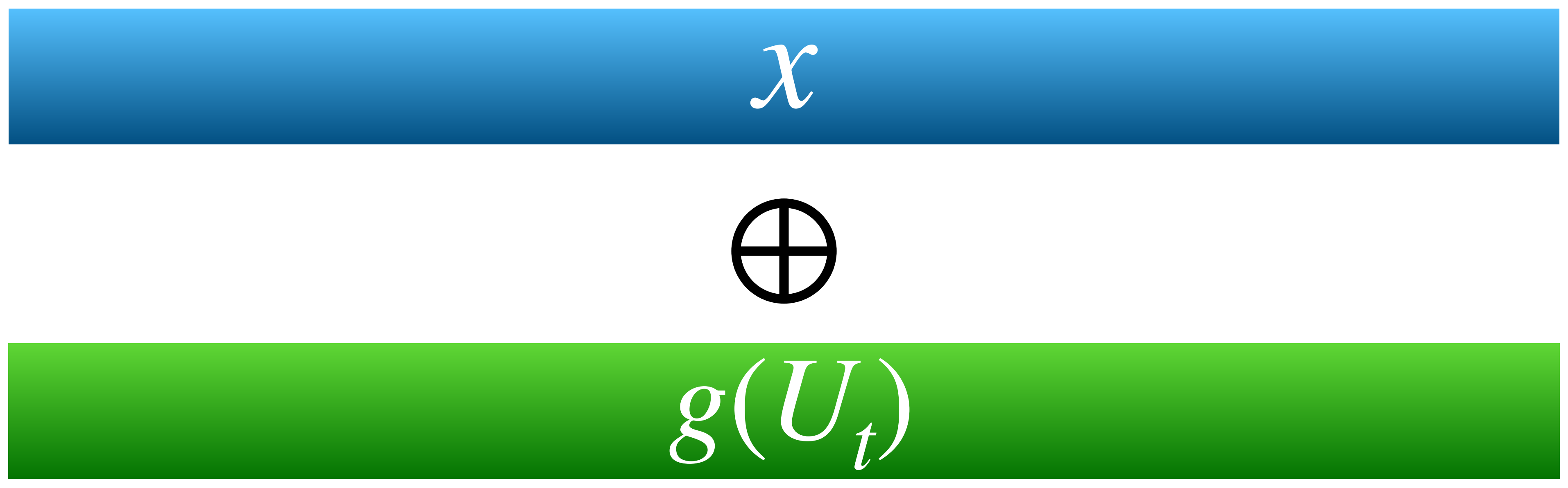

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To encode $x \in \{0,1\}^m$:

x satisfies property whp

$\oplus$

$g(U_t)$

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$\oplus$

$g(z)$ — So there is **good** fixing $U_t = z$ that enforces property

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To encode $x \in \{0,1\}^m$**:**

$$\text{Enc}(x) = \underbrace{z}_{\log m} \,\big|\, x \oplus g(z)$$

(z chosen so that property is satisfied)

PART II:

Markers +

modified average-case trace reconstruction

Subsequence-unique strings

[HolensteinMitzenmacherPanigrahyWieder08]

A string x is w**-subsequence-unique** if no substring of length w can be obtained by deleting bits of another substring of length $1.1w$ (except for trivial containment).

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Key observation: an ϵ-almost k-wise independent string is w-subsequence-unique with high probability for decent parameters k and ϵ.

Better codes for small deletion probability

Use masking to encode blocks into "almost" subsequence-unique strings

$x \in \{0,1\}^m$ $\xrightarrow{\text{encoded as}}$ z | $x \oplus g(z)$

$\log m$ w-subsequence-unique

$w = O(\log m)$

Better codes for small deletion probability

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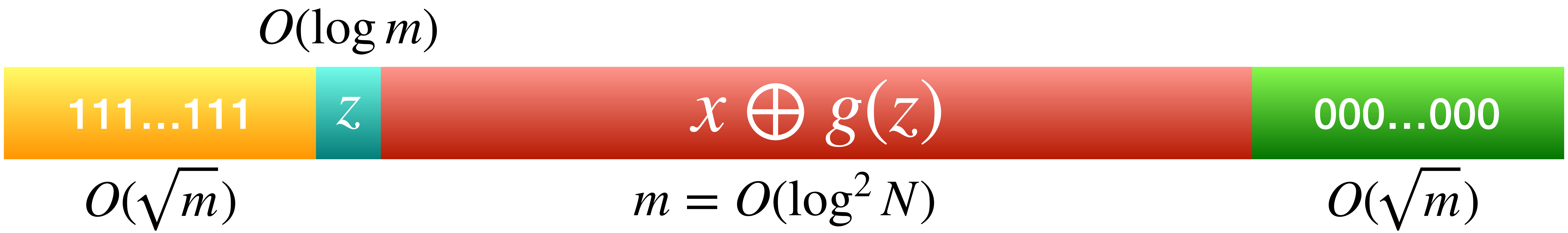
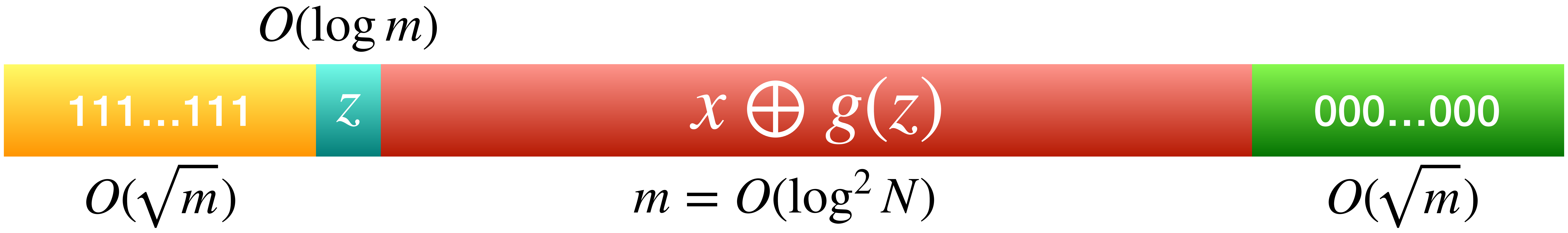


HMPW trace reconstruction algorithm doesn’t work for strings like this…

Better codes for small deletion probability



First barrier: HMPW algorithm needs to know first $\log m$ bits of $x \oplus g(z)$ "for free"

Original bootstrapping procedure requires $2^{\sqrt{m}} =$ **poly**(N) traces in this case…

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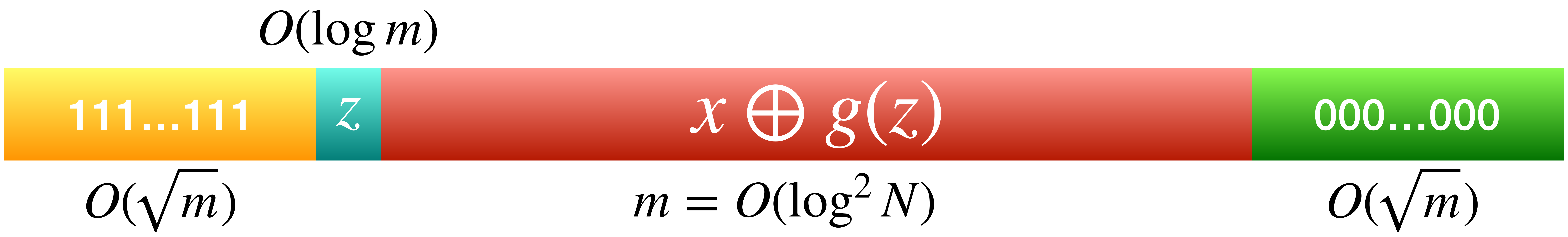

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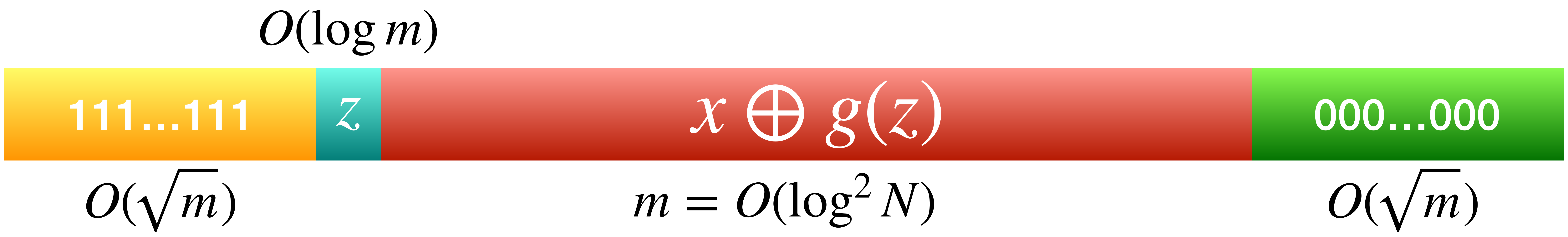
Exploit properties of masking:

$$x' = \color{red}{00\ldots00}\,||\,x \xrightarrow[\textbf{encoded as}]{} z\,||\,(\color{red}{x'} \oplus g(z))$$

Better codes for small deletion probability



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encoded as

$z' =$ systematic encoding of $0 \,||\, z$ robust against constant fraction of deletions+insertions

Better codes for small deletion probability

$O(\log m)$

111…111	z	$x \oplus g(z)$	000…000
$O(\sqrt{m})$		$m = O(\log^2 N)$	$O(\sqrt{m})$

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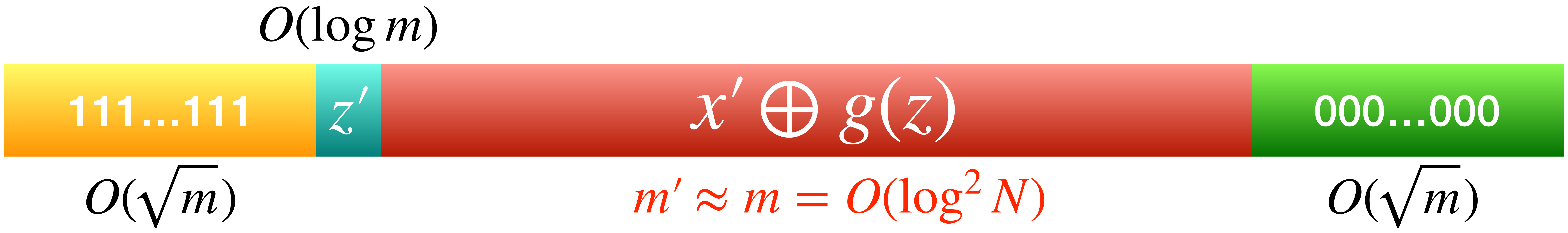

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$O(\log m)$

111…111	z'	$x' \oplus g(z)$	000…000
$O(\sqrt{m})$		$m' \approx m = O(\log^2 N)$	$O(\sqrt{m})$

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Bootstrapping only requires $O(m)$ traces now!

Better codes for small deletion probability

A few more things:

- Actually need a stronger property than subsequence-uniqueness because of markers;
- Need to make sure trace splitting works.

Almost k-wise independent string satisfies both of them with high probability.

There is an absolute constant $d^\star \in (0,1)$ such that for all $d \leq d^\star$ there exists an efficiently encodable/reconstructable code with rate $1 - O(1/\log N)$ using **polylog**(N) traces against i.i.d. deletions with probability d.

Coded trace reconstruction over large alphabets

[BrakensiekLiSpang19]

There exist efficiently encodable/reconstructable codes with rate $1-\epsilon$ over an alphabet of size $O_\epsilon(1)$ using $O(\log_{1/d}(1/\epsilon))$ traces.

Moreover, **this is tight.**

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Lower bound:

Capacity of T-use deletion channel $\leq$ Capacity of T-use erasure channel $= 1-d^T$

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Lower bound:

Capacity of T-use deletion channel $\leq$ Capacity of T-use erasure channel $= 1-d^T$

$$\implies 1-\epsilon \leq 1-d^T \implies T \geq \log_{1/d}(1/\epsilon)$$

A tool for the upper bound: Synchronization strings

[HaeuplerShahrasbi17]

A string S is a τ-**synchronization string** if for all $i < j < k$

$$ED(S_{[i,j)}, S_{[j,k)}) > (1 - \tau)(k - i)$$

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"close" to maximal edit distance everywhere

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“close” to maximal edit distance everywhere

For every τ, can efficiently construct a τ-synch-string of length N over an alphabet of size **poly**$(1/\tau)$.

Why are synchronization strings useful?

$$c = (c_1, c_2, \ldots, c_N) \in \Sigma^N \xrightarrow[]{\textbf{naive indexing}} ((c_1,1), (c_2,2), \ldots, (c_N, N)) \in (\Sigma \times [N])^N$$

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$$(c_1,1) \quad (c_2,2) \quad (c_3,3) \quad (c_4,4)$$

$$\downarrow$$

$$(c_1,1) \quad \color{red}{(c_2,2)} \quad (c_3,3) \quad \color{red}{(c_4,4)}$$

$$\Big\downarrow \ \color{red}{\textbf{deletions}}$$

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$$\downarrow \text{ deletions}$$

$$(c_1,1) \quad (c_3,3)$$

equivalent to

$$c_1 \quad c_2 \quad c_3 \quad c_4$$

$$\downarrow$$

$$c_1 \quad c_2 \quad c_3 \quad c_4$$

$$\downarrow \text{ erasures}$$

$$c_1 \quad ? \quad c_3 \quad ?$$

Why are synchronization strings useful?

[HaeuplerShahrasbi17]: *"Can efficiently transform deletions into (worst-case) erasures by indexing with synchronization string, with* ***constant*** *alphabet size blowup"*

$$S \in \Sigma^N \xrightarrow{dN \text{ deletions}} S' \in \Sigma^{(1-d)N}$$

τ-synchronization string

Why are synchronization strings useful?

[HaeuplerShahrasbi17]: *"Can efficiently transform deletions into (worst-case) erasures by indexing with synchronization string, with **constant** alphabet size blowup"*

$$S \in \Sigma^N \xrightarrow{dN \text{ deletions}} S' \in \Sigma^{(1-d)N}$$

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[HaeuplerShahrasbi17]:

There is efficient error-free indexing algorithm with at most

$$\frac{\tau dN}{1-\tau}$$

misdecodings.

Coded trace reconstruction from synchronization strings

[BrakensiekLiSpang19]: Efficient code over alphabet of size $O_\epsilon(1)$ using $O(\log_{1/d}(1/\epsilon))$ traces

S $(\tau = \epsilon^C)$ -synchronization string over alphabet of size $O_\epsilon(1)$

$\mathscr{C}$ rate $\approx 1 - \epsilon$ code robust against $\approx \epsilon^3$ fraction of erasures

Construction: *Index codewords of* $\mathscr{C}$ *with* S

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$(c_1, \ldots, c_N) \in \mathscr{C} \longrightarrow \boxed{(c_1, S_1)} \ \boxed{(c_2, S_2)} \ \cdots \ \boxed{(c_N, S_N)}$

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Reconstruction:

Trace of codeword	(x_{11}, x_{12})	(x_{21}, x_{22})	(x_{31}, x_{32})	(x_{41}, x_{42})	$\cdots$

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i_1 ? i_3 ?

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$\boxed{i_1}$ $?$ $\boxed{i_3}$ $?$

recover $c_{i_1} = x_{11}$

treat as erasure

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$\mathscr{C}$ **corrects this many erasures!**

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From large alphabets to binary codes

Construction: *Concatenate good code over large alphabet with marker-based inner code*

x N**-bit string**

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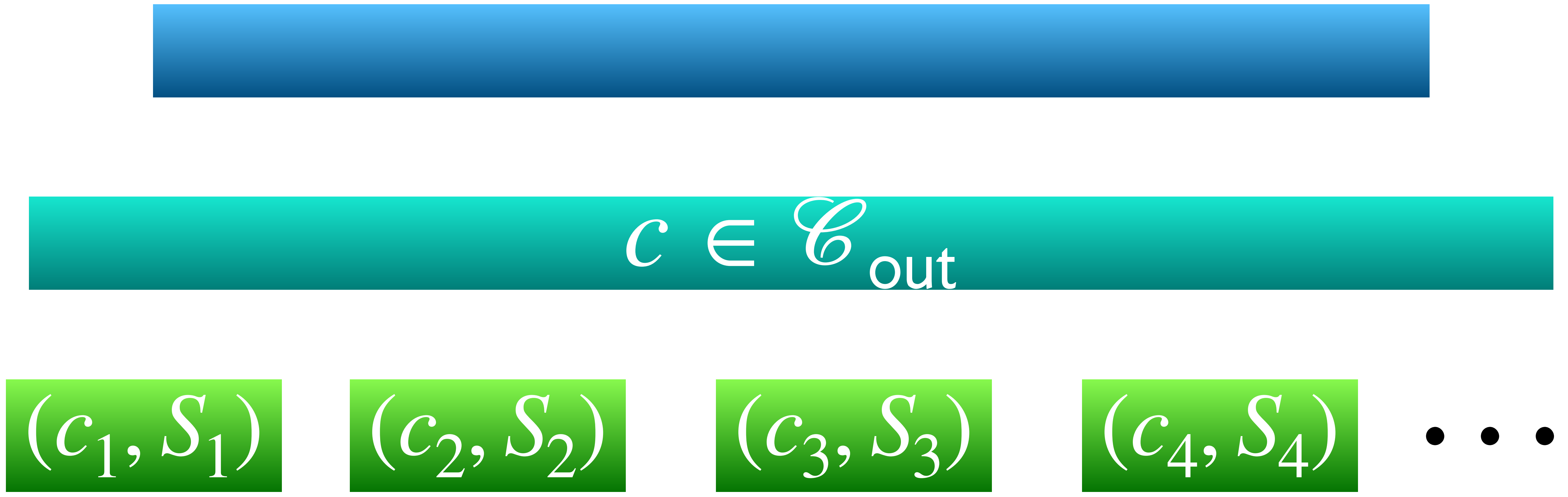

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$c \in \mathscr{C}_{\text{out}}$

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concatenate with binary code

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(c_i, S_i) $\rightarrow$ $\mathrm{Enc}_R(c_i) = $ $\underbrace{0}_{\log m}\ \underbrace{\overline{c_i}}_{m}\ \underbrace{1}_{\log m}$

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Robust against deletions, with two possible runlengths, same number of runs

Reconstruction

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Long runs **only come from markers** $\Longrightarrow$ Find traces of (most) synch symbols

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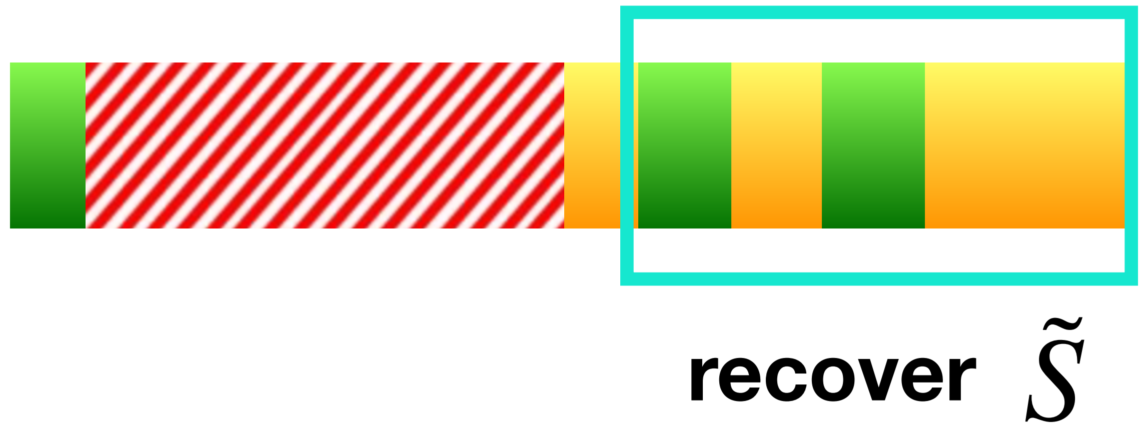

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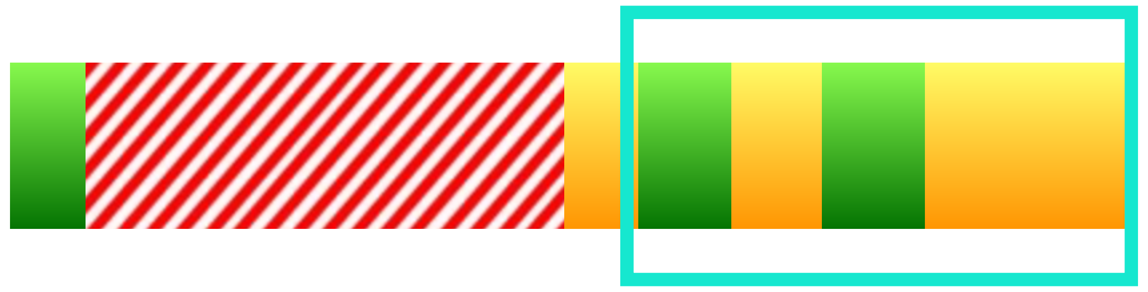

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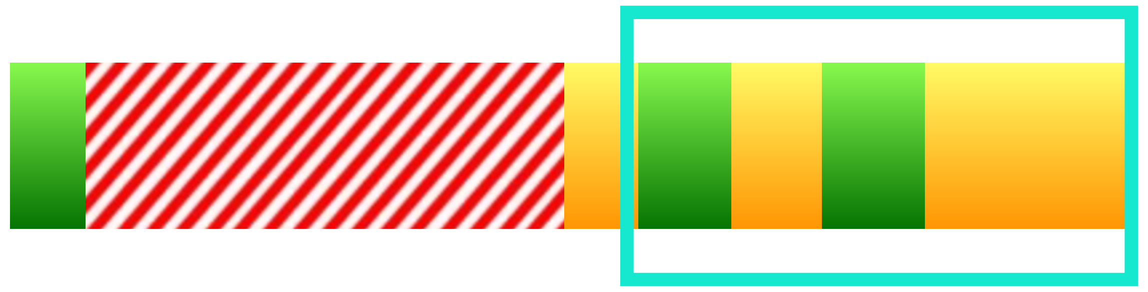

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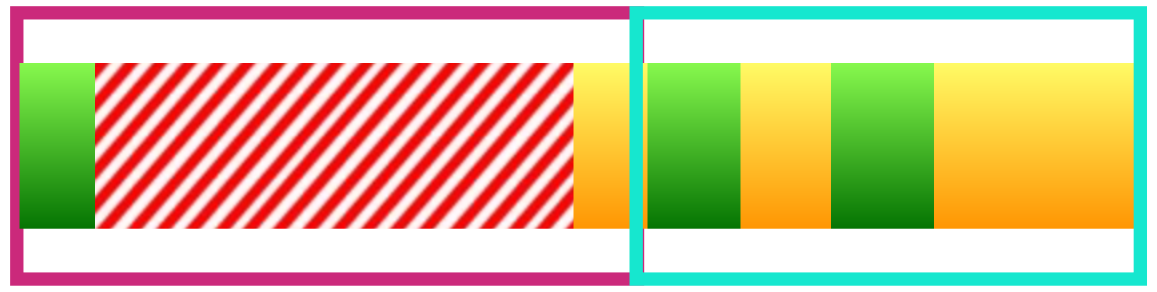

trace of c_i **whp** — **recover** $\tilde{S}$ $\longrightarrow$ **run error-free indexing algo** $\longrightarrow$ **Recover index** i

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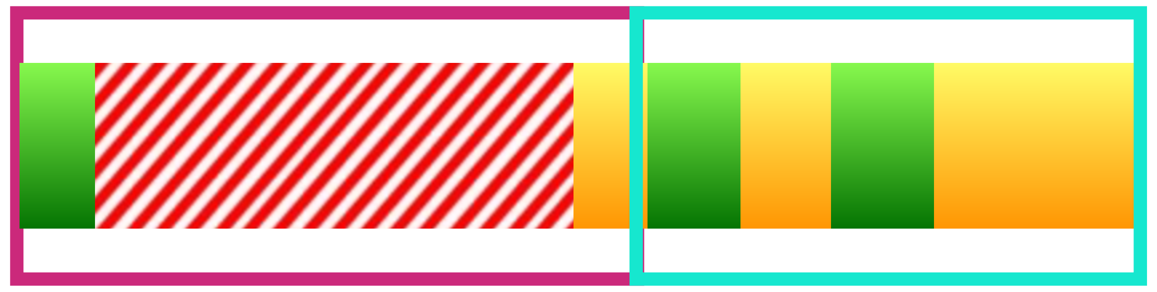

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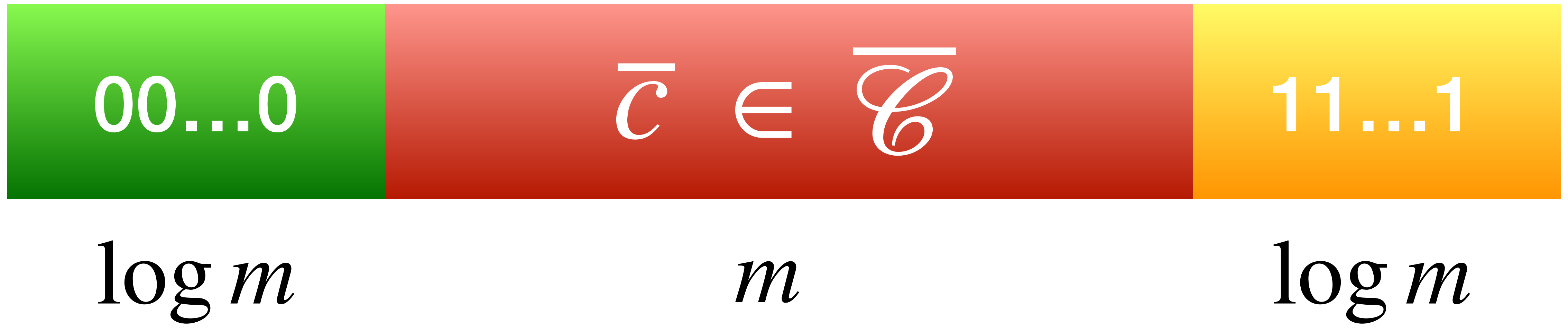
2) Use special trace rec properties of $\overline{\mathscr{C}}$ + error-correction properties of $\mathscr{C}_{\text{out}}$

Designing the (existential) inner code



Key observation: Since markers are short, **nearly all strings of this type are good for any average-case trace rec algorithm!**

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00…0	$\bar{c} \in \overline{\mathscr{C}}$	11…1
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$\Longrightarrow$ Using [HPP18], reconstruct $\overline{\mathscr{C}}$ with $\exp(\log^{1/3} m)$ traces.

Combining everything

Set inner code length to $m = \frac{1}{\epsilon} \log\left(\frac{1}{\epsilon}\right)$ to ensure rate $\geq 1 - \epsilon$

[BrakensiekLiSpang19]

For any constant deletion probability and every ϵ, there exists a code of rate $1 - \epsilon$ that can be reconstructed from $\exp(\log^{1/3}(1/\epsilon))$ traces.

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Caveat: Inner code construction takes time $\exp(m)$.

$\Longrightarrow$ Overall code construction only efficient for $\epsilon \geq \frac{\log \log N}{\log N}$

Summing up…

We can exploit worst-case and average-case trace reconstruction to design efficient high-rate codes requiring significantly fewer traces, or satisfying other nice properties.

	rate = 1 - ε	#traces	efficient encoding & reconstruction	observations
Markers + modified avg-case trace rec [CGMR19]	$\epsilon = \frac{1}{\log N}$	$\log^C N$	✓	Deletion probability smaller than absolute constant
Synch string + Markers + existential trace rec [BLS19]	$\epsilon = \frac{1}{\log N}$	$\exp((\log\log N)^{1/3})$	✓	Code construction is not efficient
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Lower bound [BLS19]: Arbitrary code of rate $1-\epsilon$ $\Longrightarrow$ requires $\approx \log^{5/2}\left(\frac{1}{\epsilon}\right)$ traces

Future work

- Efficient high-rate codes using even fewer traces;
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