On the Minimum Number of Transmissions in Single-Hop Wireless Coding Networks

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***Abstract*— The advent of network coding presents promising opportunities in many areas of communication and networking. It has been recently shown that network coding technique can significantly increase the overall throughput of wireless networks by taking advantage of their broadcast nature. In wireless networks, each transmitted packet is broadcasted within a certain area and can be overheard by the neighboring nodes. When a node needs to transmit packets, it employs the *opportunistic coding* approach that uses the knowledge of what the node's neighbors have heard in order to reduce the number of transmissions. With this approach, each transmitted packet is a linear combination of the original packets over a certain finite field.**

In this paper, we focus on the fundamental problem of finding the optimal encoding for the broadcasted packets that minimizes the overall number of transmissions. We show that this problem is NP-complete over $GF(2)$ and establish several fundamental properties of the optimal solution. We also propose a simple heuristic solution for the problem based on graph coloring and present some empirical results for random settings.

I. Introduction

In recent years, there has been an enormous interest in the design and deployment of wireless networks. Such networks are indispensable for providing ubiquitous network coverage and have many applications in both civil and military areas.

Recently, it was observed that the broadcast nature of wireless networks can be exploited in order to increase throughput and reduce energy consumption. In a wireless environment, each packet is broadcasted within a small neighborhood, which allows the neighboring nodes to overhear packets sent by their neighbors. When a node needs to transmit packets, it can employ the *opportunistic coding* [1], [2] approach that uses the knowledge of what the node's neighbors have heard in order to reduce the number of transmissions. With this approach, each transmitted packet is a linear combination of the original packets over a certain finite field.

Example 1: Consider the network depicted in Figure 1. In this example, the central node, referred to as a server, needs to deliver four packets $p_1, \dots, p_4$ to four clients $c_1, \dots, c_4$; packet p_i needs to be received by client c_i. Each client c_i has an access to some of the packets overheard from prior transmissions. This set is referred to as its "has" set. It is easy to verify that all clients can be satisfied by broadcasting two packets $p_1+p_2+p_3$ and p_1+p_4 (all additions are over $GF(2)$). Since without network coding all packets $p_1, \dots, p_4$ are needed to be transmitted, network coding allows to reduce the number of transmissions by 50%.

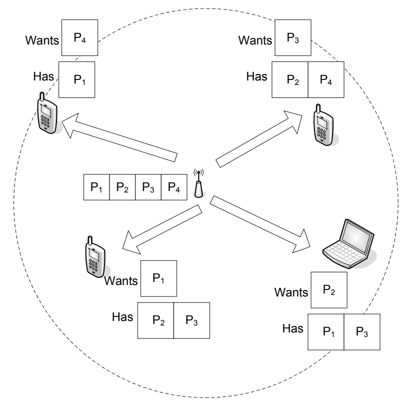

Fig. 1. Broadcast coding network

In this paper, we focus on the single hop wireless setting and consider the problem of minimizing the number of broadcast transmissions necessary to satisfy all the clients. Our contributions can be summarized as follows. First, we prove that the problem of determining the minimum number of transmissions over $GF(2)$ is NP-complete. Next, we show that the number of transmissions may depend on the size of the finite field, and that such a dependence is not necessarily monotonic. Further, we prove that the problem of finding the size of the finite field which results in the minimum number of transmissions is an NP-hard problem. Next, we establish lower and upper bounds on the *coding advantage*, i.e., the ratio between the total number of packets and the minimum number of transmissions that can be achieved by using network coding. In particular, we show that the coding advantage depends on the size of the"*has*" sets. Next, we evaluative the value of coding advantage in random settings. Finally, we present a heuristic solution based on graph coloring and verify its performance through simulations.

The considered problem is a special case of the general *network coding* [3] problem for non-multicast networks. The general network coding problem has recently attracted a large body of research (see e.g., [4], [5] and references therein), however, many of the results (such as NP-hardness) cannot be immediately extended to our problem.

While we present our results in the context of wireless data transmission, the considered problem is very general and can arise in many other practical settings. For example, consider a content distribution network that needs to deliver a set of large files (such as video clips) to different clients. In this setting, if some of the files are already available for some clients, the distribution can be efficiently implemented by multicasting a (small) set of linear combinations of the original files.

II. MODEL

We consider a one-hop wireless channel with a single server s and a set of m clients $C = \{c_1, \ldots, c_m\}$. The server needs to transmit a set $P = \{p_1, p_2, \ldots, p_n\}$ of packets to the clients. Each client requires a certain subset of packets in P, while some packets in P are already available to it. Specifically, each client $c_i \in C$ is associated with two sets:

- $W(c_i) \subseteq P$ - the set of packets required by c_i.
- $H(c_i) \subseteq P$ - the set of packets available at c_i;

We refer to $W(c_i)$ and $H(c_i)$ as the "wants" and "has" sets of c_i, respectively. The server can transmit any packet from P as well as linear combinations (over $GF(q)$) of packets in P. Each transmission i is specified by an encoding vector $g_i = \{g_i^j\} \in GF(q)^n$ such that the packet x_i transmitted in communication round i is equal to $x_i = \sum_{j=1}^n g_i^j \cdot p_j$. The practical issues related to this model are discussed in [2].

Our goal is to find the set of encoding vectors $\Phi = \{g_i\}$ of minimum cardinality that allow each client to decode the packets it requested. We refer to this problem as Problem *MIN-T-q*.

Problem MIN-T-q: Find the minimum number of transmissions and the corresponding set Φ of encoding vectors $\{g_i\}$, $g_i = \{g_i^j\} \in GF(q)^n$, that allow each client $c_i \in C$ to decode all the packets in its "wants" set $W(c_i)$.

We assume, without loss of generality, that for each packet $p_i \in P$, there exists at least one client $c_j \in C$ such that p_i belongs to the "wants" set $W(c_j)$ of c_j. We also assume that for each client $c_i \in C$ it holds that $H(c_i) \cap W(c_i) = \emptyset$.

Observation 2: Without loss of generality, we can assume that the "wants" set $W(c_i)$ of each client $c_i \in C$ contains exactly one packet. Indeed, we can substitute each client $c_i \in C$ whose "wants" set includes more than one packet by multiple clients $C_i = \{c_i^1, c_i^2, \ldots\}$ such that the "has" sets of all clients in C_i are equivalent to that of c_i and each client in C_i requests one of the packets in $W(c_i)$. It is easy to verify that the resulting instance of Problem *MIN-T-q* is equivalent to the original one.

III. HARDNESS RESULTS

In this section we focus on the case in which the encoding is performed over $GF(2)$ and prove that Problem *MIN-T-2*, which is a special case of Problem *MIN-T-q* for $GF(2)$, is NP-complete.

Theorem 3: Problem *MIN-T-2* is NP-complete.

Proof: It is easy to verify that the problem belongs to NP. To prove that the problem is NP-complete we show a reduction from the minimum vertex cover problem. In this problem we are given a graph $G(V,E)$ and need to find a subset $\hat{V}$ of V, of minimum cardinality, such that each edge $e \in E$ is incident to at least one of the nodes in $\hat{V}$. We denote by $OPT^{VC} = |\hat{V}|$ the size of the optimal solution for the vertex cover problem.

Given an instance $G(V,E)$ to the vertex cover problem we build the following instance for Problem *MIN-T-2*. The packet set P includes a packet p_v for any node in V and a packet p_e for any edge in E. We denote by $P_V = \{p_v \mid v \in V\}$ the subset of packets in P that correspond to nodes in V and by $P_E = \{p_e \mid e \in E\}$ the subset of packets in P that correspond to edges in E.

For each edge $e(v,u) \in E$ we define two clients c_e^1 and c_e^2 such that:

- $H(c_e^1) = \{p_e\}$ and $W(c_e^1) = \{p_v, p_u\}$;
- $H(c_e^2) = \{p_v, p_u\}$ and $W(c_e^2) = \{p_e\}$.

We denote by OPT the size of the optimal solution for this instance of Problem *MIN-T-2*, i.e., the minimum number of transmissions necessary to satisfy all clients. In the following two lemmas we prove that $OPT = OPT^{VC} + |E|$.

Lemma 4: $OPT \leq OPT^{VC} + |E|$.

Proof: Let $\hat{V} \subseteq V$ be the optimal solution to the vertex cover problem. Then, all clients can be satisfied by transmitting the following set of packets of size $OPT^{VC} + |E|$:

1) For each node $v \in \hat{V}$ we transmit the corresponding packet p_v;
2) For each edge $e(v,u) \in E$ we transmit the packet $p_v + p_u + p_e$, where p_v, p_u, and p_e are packets that correspond to nodes v, u, and edge e, respectively.

It is easy to verify that the set Φ of corresponding encoding vectors is a feasible solution to Problem *MIN-T-2*. Since the total number of transmitted packets is $OPT^{VC} + |E|$ it follows that $OPT \leq OPT^{VC} + |E|$. ■

Lemma 5: $OPT \geq OPT^{VC} + |E|$.

Proof: Consider an optimal solution $\Phi = \{g_1, \ldots, g_{OPT}\}$ to Problem *MIN-T-2*, where

$$g_i = (g_i^{v_1}, \ldots, g_i^{v_{|V|}}, g_i^{e_1}, \ldots, g_i^{e_{|E|}}) \in GF(2)^{|E|+|V|}.$$

With this solution, the packet transmitted at round i is equal to

$$x_i = \sum_{v_j \in V} g_i^{v_j} \cdot p_{v_j} + \sum_{e_j \in E} g_i^{e_j} \cdot p_{e_j}.$$

We denote by $\langle\Phi\rangle$ the linear subspace of dimension OPT of $GF(2)^{|V|+|E|}$ generated by the vectors in Φ.

We show that there exist two sets Φ_1 and Φ_2 of vectors in $\langle\Phi\rangle$ and a vertex cover $\hat{V} \subseteq V$ such that the following three conditions holds:

(1) For any edge $e \in E$, there exists an encoding vector $g_i \in \Phi_1$ such that $g_i^e = 1$, and $g_i^{e'} = 0$ for any edge $e' \in E \setminus \{e\}$;
(2) For each $g_i \in \Phi_2$ it holds that $g_i^e = 0$ for any edge $e \in E$;
(3) For each $v \in \hat{V}$ there exists an encoding vector $g_i \in \Phi_2$ such that $g_i^v = 1$ and $g_i^{v'} = 0$ for any node $v' \in \hat{V} \setminus \{v\}$.

Note that all encoding vectors in $\Phi_1 \cup \Phi_2$ are linearly independent, $|\Phi_1| = |E|$, and $|\Phi_2| = |\hat{V}|$.

First, we show how to construct the set Φ_1. Let $e(v,u)$ be an edge in E and let c_e^1 and c_e^2 be the two clients that correspond to e. We note that in order to satisfy c_e^2, $\langle\Phi\rangle$ must contain at least one vector g_i for which it holds $g_i^e = 1$ and $g_i^{e'} = 0$ for any edge $e' \in E \setminus \{e\}$. Thus, we can form Φ_1 by including, for each $e \in E$, the vector $g_i \in \langle\Phi\rangle$ that corresponds to e.

Second, we show how to construct set Φ_2 and the vertex cover $\hat{V}$. Again, let $e(v,u)$ be an edge in E and let c_e^1 and c_e^2 be the two clients that correspond to e. Note that, in order to satisfy the client c_e^1, the set $\langle\Phi\rangle$ must contain a vector g_i for which it holds that $g_i^{e'} = 0$ for all $e' \in E$, $g_i^w = 0$ for all $w \in V \setminus \{v,u\}$, and either g_i^v or g_i^u (or both) are non-zero. Let T be a set that contains such vectors for all $e \in E$. Let $l = \dim\langle T\rangle$. It follows from linear algebra that there exists an $l \times (|V|+|E|)$ matrix M over $GF(2)$ that satisfies the following conditions:

1) The rows of M span $\langle T\rangle$;
2) There are l linearly independent columns in M such that each column contains exactly one non-zero element.

Indeed, we can first construct an $l \times (|V|+|E|)$ matrix M' whose rows span T. Such matrix is of rank l, hence it contains at least l non-zero columns which are linearly independent. The matrix M can be constructed form M' by performing Gaussian elimination. We denote by $\hat{V}$ the subset of V that corresponds to l linearly independent columns of M, each column contains exactly one non-zero element. Then, we set Φ_2 to be the set of row vectors of M. Note that Φ_2 has l elements.

We proceed to show that $\hat{V}$ is a vertex cover in $G(V,E)$. We note that the structure of M implies that for any non-zero vector g_i in the row span of M, and, in turn, in $\langle T\rangle$ it must hold that $g_i^w = 1$ for some $w \in \hat{V}$. For each edge $e(v,u) \in E$ let g_i be the vector that correspond to e in T. Recall g_i has one or two non-zero components, which are either g_i^v or g_i^u, or both. This implies that either v or u, or both belong to $\hat{V}$.

We proved that there exist two sets Φ_1 and Φ_2 of independent vectors in $\langle\Phi\rangle$ such that $|\Phi_1| = |E|$, and $|\Phi_2| \geq OPT^{VC}$. We conclude that

$$OPT = \dim\langle\Phi\rangle \geq |\Phi_1| + |\Phi_2| \geq OPT^{VC} + |E|.$$

■

From lemmas 4 and 5 it follows that $OPT = OPT^{VC} + |E|$. Thus, a polynomial-time algorithm that solves Problem *MIN-T-2* will solve the vertex cover problem as well, resulting in a contradiction. ■

IV. Dependence on the Field Size

In this section we consider a variant of Problem *MIN-T-q* which allows flexibility in choosing the underlying finite filed. $GF(q)$. Specifically, for each instance of the problem, we can choose the finite field that minimizes the required number of transmissions.

We denote by $OPT(q)$ the minimum required number of transmissions over $GF(q)$. We also denote by OPT the minimum number of transmissions that can be achieved over any finite field.

We begin by observing that the minimum number of transmissions may depend on the size of the finite field $GF(q)$. For example, consider the problem described in Table I, where $P = \{p_1, p_2, p_3, p_4\}$, and for every client c_i, $H(c_i) = P \setminus W(c_i)$. We prove that in this problem $OPT(2) > OPT(3)$.

First, we show that $OPT(2) > 2$. Suppose, by way of contradiction, that there is a solution to this problem with two transmissions:

$$x_1 = g_1^1 p_1 + \cdots + g_1^4 p_4$$
$$x_2 = g_2^1 p_1 + \cdots + g_2^4 p_4$$

To satisfy all clients, the vectors $(g_1^1, g_2^1), \ldots, (g_1^4, g_2^4)$ should be all distinct and different from $(0,0)$, which is not possible over $GF(2)$. Since the set of transmissions $\{p_1+p_3, p_2+p_3, p_4\}$ satisfies all clients, it follows that $OPT(2) = 3$. We note that OPT is at least two, since $OPT \geq |W(c_i)| = 2$. We also observe that over $GF(3)$ only two transmissions $\{p_1+p_3+p_4, p_2+p_3+2p_4\}$ are sufficient, hence $OPT(3) = OPT = 2$.

C	$W(c_i)$	$H(c_i)$
c_1	$\{p_1, p_2\}$	$\{p_3, p_4\}$
c_2	$\{p_1, p_3\}$	$\{p_2, p_4\}$
c_3	$\{p_1, p_4\}$	$\{p_2, p_3\}$
c_4	$\{p_2, p_3\}$	$\{p_1, p_4\}$
c_5	$\{p_2, p_4\}$	$\{p_1, p_3\}$
c_6	$\{p_3, p_4\}$	$\{p_1, p_2\}$

TABLE I

The following lemma shows that $OPT(q)$ is not necessarily a monotonic function of q.

Lemma 6: There exists an instance of Problem *MIN-T-q* for which it holds that $OPT(q) = 3$ for fields with odd characteristic, such as $GF(3)$ and $OPT(q) > 3$ for fields with even characteristic.

Proof (sketch): Consider the problem instance described in Table II, where $P = \{p_1, \ldots, p_7\}$, and $H(c_i) = P \setminus W(c_i)$.

c_i	$W(c_i)$
c_1	$\{p_1\}$
c_2	$\{p_2\}$
c_3	$\{p_3\}$
c_4	$\{p_2, p_4\}$
c_5	$\{p_3, p_5\}$
c_6	$\{p_3, p_6\}$
c_7	$\{p_4, p_7\}$
c_8	$\{p_5, p_7\}$
c_9	$\{p_6, p_7\}$
c_{10}	$\{p_4, p_5, p_6\}$

TABLE II

For fields with odd characteristic, the transmission sequence

$$\{p_1 + p_4 + p_5 + p_7, p_2 + p_4 + p_6 + p_7, p_3 + p_5 + p_6 + p_7\}$$

satisfies all clients, hence $OPT(q) = OPT = 3$.

We observe that for fields with even characteristic ($q = 2^k$) it holds that $OPT(2^k) > 3$. Indeed, for any solution $\{g_1, g_2, g_3\} \in GF(q)^3$ with three transmissions, consider the matrix T whose row vectors are g_1, g_2, and g_3. To satisfy the demands of all the clients the vector matroid of T should be isomorphic to the Fano matroid [6]. But, the Fano matroid is only representable over fields with odd characteristics. Therefore, there are no solutions to above problem with three transmissions over $GF(2^k)$. ■

The next lemma shows that deciding whether the optimal number of transmissions can be achieved for a given field $GF(q)$ is an NP-hard problem.

Lemma 7: For given a prime power q, it is an NP-hard problem to decide whether $OPT(q) = OPT$.

Proof (sketch): Similar to [5], we use a reduction from the problem of graph coloring. Given an undirected graph $G(V, E)$, we construct the following instance to the broadcast problem. For each node $v \in V$, the set P includes a packet p_v. For each edge $e(v, u) \in E$, the set C includes a client c_e such that $W(c_e) = \{p_v, p_u\}$ and $H(c_e) = P \setminus W(c_e)$. It is easy to verify that for this problem it holds that $OPT = 2$.

We show the problem can be solved with two transmissions over $GF(q)$ if and only if G is $q+1$ colorable. First, suppose that $G(V, E)$ can be colored with $q + 1$ colors. Let $d(v) \in \{1, \ldots, q+1\}$ be the color of vertex v. As shown in [5], there exists $q+1$ pairwise independent vectors $(z_1^1, z_2^1), \ldots, (z_1^{q+1}, z_2^{q+1})$ over $GF(q)$. For each node $v \in V$ we set $(g_1^v, g_2^v) = (z_1^{d(v)}, z_2^{d(v)})$. It is easy to verify that the two encoding vectors $(g_1^v)_{v \in V}$ and $(g_2^v)_{v \in V}$ constitute a feasible solution for the broadcast problem.

Second, suppose that there exists a solution $\Phi = \{(g_1^v)_{v \in V}, (g_2^v)_{v \in V}\}$ for the broadcast problem with two transmissions. We show that this implies that there exists a $q+1$ coloring of graph G. For each vertex $v \in V$, the vector (g_1^v, g_2^v) determines the coefficients for packet p_v for the first and the second transmissions in Φ. The set of such vectors can be partitioned into $q+1$ equivalence classes, such that any two linearly dependent vectors are placed into the same equivalence class. Next, for each equivalence class we assign one of the $q + 1$ colors. Next, for each vertex $v \in V$ we assign the color that corresponds to the equivalence class of (g_1^v, g_2^v). It is easy to verify that this will result in a valid coloring of G that requires at most $q + 1$ colors. ■

V. BOUNDS ON CODING ADVANTAGE

Given a one-hop transmission problem with n packets $P = \{p_1, p_2, \ldots, p_n\}$ and m clients $C = \{c_1, \ldots, c_m\}$, we define the coding gain Γ as the ratio between the minimum number of transmissions without coding and the minimum number of transmissions with coding, i.e.,

$$\Gamma = \frac{n}{OPT},$$

where OPT is the minimum number of transmissions achievable over any finite field $GF(q)$.

Let $L = \max_{c_i \in C} |H(c_i)|$ and $\ell = \min_{c_i \in C} |H(c_i)|$. The following theorem establishes lower and upper bounds on Γ.

Theorem 8: The coding gain is bounded by

$$\frac{n}{n - \ell} \leq \Gamma \leq L + 1 \qquad (1)$$

Proof: We assume, without loss of generality, that $|W(c_i)| = 1$ for each client $c_i \in C$. Let $GF(q)$ be the field that requires OPT transmissions. Consider an optimum solution Φ that includes OPT encoding vectors $g_1, \ldots, g_{OPT}$.

Let $e_j \in GF(q)^n$, $1 \leq j \leq n$, be the unit vector whose components are all zeros except for the j-th one which is 1. For $1 \leq i \leq n$, we define $w_i = e_j$ if client c_i wants packet p_j. Also, we define

$$\hat{H}(c_i) = \{e_j; p_j \in H(c_i)\}.$$

To guaranty that each client c_i is able to decode the packet h_i in its "wants" list, there must be a vector $y_i \in \langle \Phi \rangle$ such that $w_i = y_i + h_i$, $h_i \in \langle \hat{H}(c_i) \rangle$, where $\langle \Phi \rangle$ and $\langle \hat{H}(c_i) \rangle$ are the linear subspaces generated by the vectors in Φ and $\hat{H}(c_i)$, respectively. We note that the Hamming weight of y_i is upper bounded by $L + 1$.

Let $Y = \{y_i \mid c_i \in C\}$. By the optimality of the solution the dimension of the linear subspace $\langle Y \rangle$ generated by Y is equal to that of $\langle \Phi \rangle$. Let B be the $OPT \times n$ matrix whose row vectors belong to Y and form the basis of Y. We note that B must satisfy the following two conditions:

1) Each row of B contains at most $L+1$ non-zero elements.
2) B does not contain the all-zero column vector.

The first condition follows from the upper bound on the Hamming weights of the vectors in Y. The second condition follows from the observation that for every packet p_i at the source, there is at least one client that wants it. These two conditions imply that $OPT \geq \frac{n}{L+1}$.

We proceed with the lower bound. Given an instance I_1 of Problem *MIN-T-q*, we form another instance I_2 with where all the "has" sets have order ℓ, and where $W(c_i) = P \setminus H(c_i)$. Instance I_2 is formed by deleting arbitrary elements from the "has" sets of I_1 and expanding the "wants" sets of its elements. Note that any valid solution for instance I_2 is also a valid solution for instance I_1.

For a field $GF(q)$ of large enough size (larger than the number of clients), we can always find a subspace S of dimension $n-\ell$ in $GF(q)^n$ that is simultaneously orthogonal to all the subspaces $<\hat{H}(c_i)>$ corresponding to I_2 (Theorem 1 in [7]). Any basis of S will constitute a solution for I_2, and, in turn, for I_1, which requires $n-\ell$ transmissions. Thus, the lower bound follows. ■

VI. Heuristic Approach and Numerical Results

A. Heuristic Approach

In Section III, we proved that Problem *MIN-T-2* is NP-complete, hence finding an optimal solution for large instances of the problem can be impractical. In this section, we present a heuristic approach to solve this problem. Our heuristic solution employs *memoryless decoding*, i.e., each client uses exactly one of the transmitted packets to decode one of the packets in its "wants" list and never uses a linear combination of the transmitted packets. While memoryless decoding, in general, results in a suboptimal solution, our numerical results, presented below, show that in many cases the number of required packets is close to the optimum. We observe that the problem of finding the minimum number of transmissions with memoryless decoding is equivalent to the problem of finding the minimum chromatic number of an undirected graph.

Specifically, consider an instance I problem of Problem *MIN-T-q*, in which the "wants" set of each client is of cardinality one. Then, we construct an instance $G(V,E)$ to graph coloring problem through the following procedure:

- For each client $c_i \in C$ there is a corresponding vertex v_{c_i} in V
- Each two vertices v_{c_i} and v_{c_j} are connected by an edge if one of the following holds:
  – Clients c_i and c_j have identical "wants" sets;
  – $W(c_i) \subseteq H(c_j)$ and $W(c_j) \subseteq H(c_i)$.

Let $\hat{V} \subseteq V$ be a clique in $G(V,E)$, i.e., each two vertices of V are connected by an edge in G. Note that all clients that correspond to nodes in $\hat{V}$ can be satisfied by one transmission, which includes a linear combination of all packets in their "wants" sets. Thus, the minimum number of transmissions with memoryless decoding can be found by solving a *clique partition* problem [8], i.e., partition of V into disjoint subsets $V_1, V_2, \ldots, V_k$, such that for $1 \leq i \leq k$, the subgraph of G induced by V_i is a complete graph. This problem, in turn, corresponds to the minimum graph coloring problem of the complimentary graph. The latter problem is a well-studied problem with a wealth of heuristic solutions developed in the recent years.

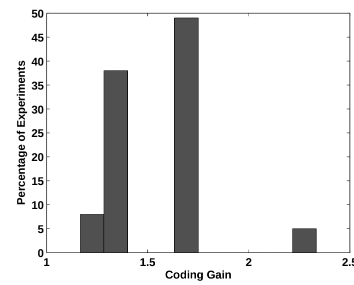

Fig. 2. Histogram of coding gain for 7 clients with optimal decoding

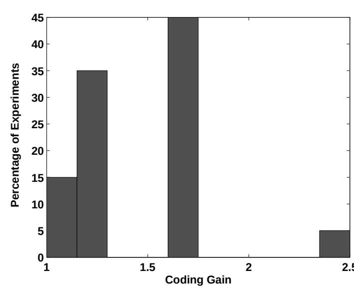

Fig. 3. Histogram of coding gain for 5 clients using memoryless decoding

B. Numerical results

We performed several numerical experiments in order to evaluate the coding gain as well as the performance of the heuristic solution in random settings. In all of our experiments described below, the "wants" set of each client is of cardinality one, and the number of clients is equal to the number of packets.

In the first experiment, we evaluated the coding gain of a single-hop wireless system with seven clients. Specifically, we generated 50 instances of Problem *MIN-T-q*, in each setting the set "has" of each client is randomly selected. The results of the experiment are shown in Figure 2. The results show that in the majority of the experiments, there is a significant coding gain (more than 1.75).

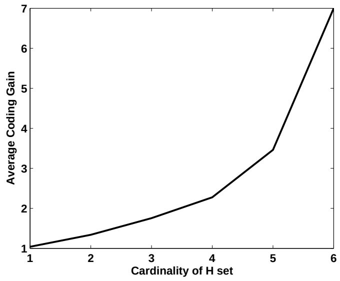

Fig. 4. Average Coding gain as a function of the cardinality of the "has" set.

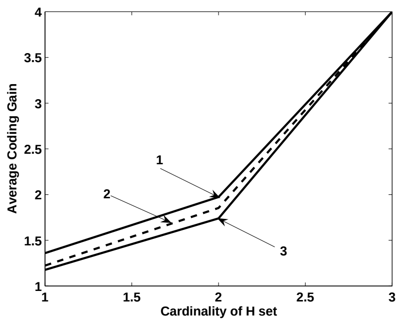

Fig. 5. Average Coding gain as a function of the cardinality of the "has" set using different techniques: (1) optimal decoding; (2) memoryless decoding; (3) heuristic approach.

The second experiment is similar to the first one, but the clients only employ memoryless decoding. The results of the experiment are shown in Figure 3. The results show that a significant coding gain (up to 2.5) can be achieved, while in the majority of the cases, the coding gain is at least 1.7.

In the third experiment, we studied the dependence of average coding gain on the cardinality of the "has" sets. In particular, we generated a problem instance in which the cardinality of the "has" set is equal for all clients, while the content of the "has" set was randomly selected. Figure 4 shows the average coding gains of the system with seven clients using optimal decoding as a function of cardinality of the "has" sets, while Figure 5 shows the comparison of average coding gains as a function of cardinality of the "has" sets, for three techniques i.e., optimal decoding, memoryless decoding and heuristic approach.The results show that the average coding gain increases with the size of the "has" sets, which confirms the intuition that coding is more beneficial if the clients have more packets in their "has" sets.

Finally, we evaluated the coding gain that can be obtained through the heuristic approach presented in Section VI-A. The results of this experiment are depicted in Figure 6. The results show that the proposed heuristic approach allows to obtain a significant reduction in the number of transmitted packets.

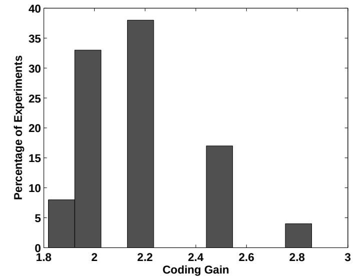

Fig. 6. Histogram of coding gain for 20 clients using heuristic approach

VII. Conclusion

The paper focuses on minimizing the number of transmissions necessary for satisfying all clients in single-hop wireless settings. We employ the technique of network coding which allows to take advantage of the packets that were overheard from prior transmissions.

Our paper makes the following contributions. First, we proved that the problem of finding the minimum number of transmissions is NP-complete over the binary field. Second, we analyzed an extended version of the problem in which the encoding can be performed over a larger finite field. Furthermore, we established lower and upper bounds on the value of the coding gain. Next, we presented a heuristic solution based on graph coloring. Finally, we conducted a simulation study that evaluates the coding gains in practical settings.

The considered problem presents significant challenges and provides a fertile ground for future research. In particular, we would like to prove the NP-hardness and inapproximability for finite fields of larger size as well as for non-linear network codes.

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