Efficient Network Coding Algorithms For Dynamic Networks

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***Abstract*—Network coding is a new paradigm that allows the intermediate nodes in a network to create new packets by combining the packets received on their incoming edges. The central problem in the design of network coding schemes is to assign local encoding coefficients for the intermediate edges in a way that allows every terminal node to decode the packets generated by the source node. The main applications of the network coding technique include content distribution, peer-to-peer networks, and wireless ad-hoc networks. Such networks are characterized by highly dynamic set of users and frequent topological changes.**

In this paper we focus on the design of efficient multicast network codes for dynamic networks. First, we consider the problem of maintaining the feasibility of a given network code upon a change in the network topology or the addition of a new user. Our goal is to minimize the number of encoding coefficients that needs to be modified to keep the network code feasible. Second, we present a new network coding algorithm that uses path-based coding assignments to efficiently handle frequent changes in the network topology and the multicast group.

I. Introduction

Network coding [1] is a new promising technique with many potential applications in networking and distributed computing. The central problem in network coding is to design an efficient algorithm that assigns the local encoding coefficients in a way that allows each terminal node to decode the required packets. Network coding has a substantial potential for improving throughput, reliability, and robustness of wired and wireless networks.

Currently, the main applications of the network coding technique are in the areas of content-distribution networks [2], peer-to-peer networks [3], and wireless networks [4]. Such networks typically have highly dynamic topologies and a frequently changing set of users. Accordingly, there is a need to develop network coding algorithms that can efficiently handle frequent network changes.

In this paper, we focus on maintaining the feasibility of a given network code upon an addition of a new user or a change in the network topology. Our goal is to minimize the number of encoding coefficients that need to be modified to keep the network code feasible. A smaller number of required changes in the encoding coefficients will allow the coding network to adjust for a new user or a change in the network topology more efficiently and also reduce the disruption for existing users. We also present a new algorithm that uses path-based assignment of encoding coefficients to construct a feasible network code. While the path-based approach typically requires a larger field size than the standard algorithms such as the one due to Jaggi et al [5], it can handle network changes through fast and efficient procedure, with the limited number of changes required in the network code.

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Related Work. Network coding research has been initiated by the seminal paper by Ahlswede et al. [1], and since then attracted a significant interest from the research community. Koetter and Médard [6] developed an algebraic framework for network coding. This framework was used by Ho et al. [7] to show that linear network codes can be efficiently constructed through a randomized algorithm. Jaggi et al. [5] proposed a deterministic polynomial-time algorithm for finding feasible network codes in multicast networks. Network coding algorithms for dynamic networks have been studied in references [8], [9], and [10]. Ho *et al.* [8] showed that the network coding approach provides substantial benefits in dynamically varying environments. Zhao and Médard [9] considered the problem of modifying network topology in a way that minimizes the number of required code rearrangements. Their algorithm uses linear programming approach and relies on a cost function that penalizes edges whose addition might require a change in the network code. Ho *et al.* [10] presented a framework for network management based on the network coding approach and considered a problem of minimizing the number of network codes required for handling all single edge failures.

II. Network model

We consider a multicast network $\mathbb{N}$ that uses a directed acyclic graph $G(V, E)$, with vertex set V and edge set E, to send data from source s to a set T of terminal nodes. The number of terminal nodes is denoted by k, i.e., $k = |T|$. The data is delivered in packets, each packet is an element of a finite field $\mathbb{F}_q = GF(q)$. We assume that the communication is performed in rounds, such that each edge $e \in E$ transmits one packet per round. Note that this does not result in any loss of generality since edges of higher capacity can be substituted by multiple parallel edges. At each communication round, the

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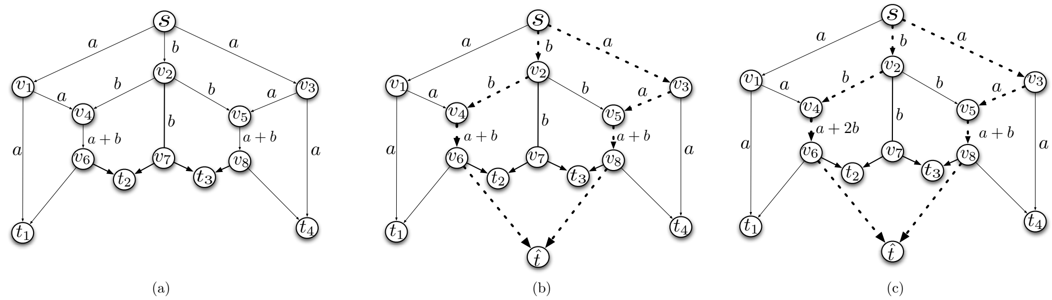

Fig. 1. (a) Original network $\mathbb{N}$ and a feasible network code $\mathbb{C}$ for $\mathbb{N}$. (b) A new network $\hat{\mathbb{N}}$ that includes a new terminal node $\hat{t}$. (c) A new network code $\hat{\mathbb{C}}$ for $\hat{\mathbb{N}}$.

source node needs to transmit h packets $\mathbb{R} = (p_1, p_2, \ldots, p_h)^T$ from the source node $s \in V$ to each destination node $t \in T$. We refer to h as the *rate* of the multicast connection. It was shown in [1] and [11] that the maximum rate of the network, i.e., the maximum number of packets that can be sent from the source s to a set T of terminals per time unit, is equal to the minimum capacity of a cut that separates the source s from a terminal $t \in T$. Accordingly, we say that a multicast network $\mathbb{N}$ is *feasible* if any cut that separates s and a terminal $t \in T$ has at least h edges.

Without loss of generality, we assume that the source node s has exactly h incoming edges, each incoming edge transmits one of the original packets in $\mathbb{R}$. We also assume that each terminal $t \in T$ also has h incoming edges and no outgoing edges. For each edge $e \in E$ we denote by p_e the packet transmitted on that edge.

Let $e(v,u)$ be an edge in E and let $\mathcal{M}_e$ be the set of incoming edges of its tail node v, $\mathcal{M}_e = \{(w,v) \mid (w,v) \in E)\}$. Then, we associate with each pair of edges $\{(e',e) | e' \in \mathcal{M}_e\}$ a *local encoding coefficient* $\beta_{e',e} \in \mathbb{F}_q$. The local encoding coefficients of the edges that belong to $\mathcal{M}_e$ determine the packet p_e transmitted on edge e as a function of packets transmitted on the incoming edges of e. Specifically, the packet p_e is equal to

$$p_e = \sum_{e' \in \mathcal{M}_e} \beta_{e',e} \cdot p_{e'}, \tag{1}$$

where all operations are performed over $\mathbb{F}_q$.

We say that edge e' is *adjacent* to edge e if the head node of e' is identical to the tail node of e. We denote by $\mathbb{S}$ the set of the adjacent pairs of edges in the network. We refer to the set of local encoding coefficients as $\mathbb{C} = \{\beta_{e',e} \mid (e',e) \in \mathbb{S}\}$ as a network code for $\mathbb{N}$.

Note that each packet transmitted over the network is a linear combination of the original packets $p_1, p_2, \ldots, p_h$ generated by the source node s. Accordingly, for each edge $e \in E$ we define the *global encoding vector* $\Gamma_e = (\gamma_1^e, \ldots, \gamma_h^e)^T \in \mathbb{F}_q^h$, that captures the relation between the packet p_e transmitted on edge e and the original packets in $\mathbb{R}$:

$$p_e = \sum_{i=1}^{h} \gamma_i^e \cdot p_i. \tag{2}$$

For each terminal t in T we define the *transfer matrix* $\mathbb{M}_t$ that captures the relation between the original packets $\mathbb{R}$ and the packets received by the terminal node $t \in T$ over its incoming edges. The matrix $\mathbb{M}_t$ is defined as follows:

$$\mathbb{M}_t = \left[\ \Gamma_{e_t^1} | \quad \Gamma_{e_t^2} | \quad \ldots \quad | \Gamma_{e_t^h}\ \right], \tag{3}$$

where $E_t = \{e_t^1, \ldots, e_t^h\}$ is the set of incoming edges of terminal t.

Our goal is to find a set of network coding coefficients $\mathbb{C} = \{\beta_{e',e} \mid (e',e) \in \mathbb{S}\}$ that allows each terminal to decode the original packets $\mathbb{R}$ from the packets obtained on its incoming edges. This can be accomplished only if the matrix $\mathbb{M}_t$ is a full-rank matrix for each terminal $t \in T$. The assignment of $\mathbb{C}$ that satisfies this condition is referred to as *feasible network code*.

III. NETWORK CODING IN DYNAMIC NETWORKS

In this paper we focus on network coding algorithms for dynamic networks. When a new user joins the network or when the network undergoes a topological change (e.g., due to a failure of an edge), the existing network code might no longer be feasible. For instance, consider the example depicted in Figure 1. The original network $\mathbb{N}$ together with the corresponding network code $\mathbb{C}$ is depicted in Figure 1(a). The network delivers two packets, a and b, to four terminals, t_1, t_2, t_3, and t_4. Suppose that a new terminal $\hat{t}$ joins the network. In order to maintain the feasibility requirement, two new edges, $(v_6, \hat{t})$ and $(v_8, \hat{t})$ have been added to the network. The resulting multicast network $\hat{\mathbb{N}}$ is depicted in Figure 1(b). Since the global encoding coefficients of edges (v_4, v_6) and (v_5, v_8) are linearly dependent, and since nodes v_6 and v_7 can only forward their incoming packets, we need to change some of the encoding coefficients for edges in $\mathbb{S}$ so the new terminal $\hat{t}$ will be able to decode the packets sent by source s.

A straightforward approach is to compute a new network code $\hat{\mathbb{C}}$ for $\hat{\mathbb{N}}$ from scratch. However, with this approach the local encoding coefficients for all pairs of edges in $\mathbb{S}$ might change. In practice, this can incur a substantial overhead, associated with determining, distributing, and changing the encoding coefficients for a large number of network nodes. However, in many cases, we only need to change a small number of encoding coefficients to make the new network feasible. For example, Figure 1(c) depicts a new network code $\hat{\mathbb{C}}$ formed from $\mathbb{C}$ by changing a single encoding coefficient ($\beta_{(v_2,v_4),(v_4,v_6)}$).

Let $\mathbb{N}$ be an original multicast network over a graph $G(V,E)$ with a set $\mathbb{S}$ of adjacent edges and a corresponding feasible network code $\mathbb{C} = \{\beta_{e',e}\}$. Let $\hat{\mathbb{N}}$ be a modified multicast network over graph $\hat{G}(\hat{V},\hat{E})$, $V \subset \hat{V}$, with set of adjacent edges $\hat{\mathbb{S}}$ and a feasible network code $\hat{\mathbb{C}}$. Then, the set of edges with modified local encoding coefficients is defined as

$$\bar{\mathbb{S}} = \left\{(e',e) \mid (e',e) \in \mathbb{S}, (e',e) \in \hat{\mathbb{S}}, \beta(e',e) \neq \hat{\beta}(e',e)\right\}.$$

We refer to $|\bar{\mathbb{S}}|$ as the number of changes of local encoding coefficients required by $\hat{\mathbb{C}}$. For example, the network code $\hat{\mathbb{C}}$ depicted in Figure 1 requires a change of only one coefficient.

Dynamic networks can also experience frequent changes in the network topology. In particular, some of the network edges may fail and new edges and nodes may be added to the network to maintain its feasibility. For example, consider the network $\mathbb{N}$ with the feasible network code $\mathbb{C}$ depicted in Figure 2(a), and assume that edge (v_2,v_7) has failed. The modified network topology is depicted in Figure 2(b). In the new topology, nodes t_2 and t_3 are connected to nodes v_6 and v_8. However, with the network code $\mathbb{C}$, the global encoding vectors of the edges (v_4,v_6) and (v_4,v_8) are linearly dependent. The new feasible network code $\hat{\mathbb{C}}$ for $\hat{\mathbb{N}}$ can be constructed by modifying the local encoding coefficient $\beta_{(v_2,v_4),(v_4,v_6)}$ for the pair of edges (v_2,v_4) and (v_4,v_6).

IV. Adding a new user to the multicast group

In this section we focus on the scenario in which a new user $\hat{t}$ joins the existing multicast group T. First, we establish lower and upper bounds on the number of modified network coding coefficients.

Let $\mathbb{N}$ be an original multicast network over the graph $G(V,E)$, with a set of terminal nodes T and a corresponding feasible network code $\mathbb{C} = \{\beta_{e',e}\}$. Let $\hat{t}$ be a new terminal node, and $\hat{\mathbb{N}}$ be a modified multicast network over graph $\hat{G}(\hat{V},\hat{E})$. We begin with the lower bound.

Lemma 1: Let $\mathbb{M}_{\hat{t}}$ be the transfer matrix for $\hat{t}$ with respect to the code $\mathbb{C}$. Then, a feasible network code $\hat{\mathbb{C}}$ for $\hat{\mathbb{N}}$ requires at least $h - \mathrm{rank}(\mathbb{M}_{\hat{t}})$ changes of the local encoding coefficients in $\mathbb{C}$.

Proof: It is sufficient to show that a change in one encoding coefficient can increase the rank of the transfer matrix by at most one. Suppose that we change the local encoding coefficient for a pair (e',e) of adjacent edges from $\beta'_{e',e}$ to $\beta''_{e',e}$. Let $\mathbb{M}'_{\hat{t}}$ and $\mathbb{M}''_{\hat{t}}$ be the transfer matrices before and after the change, respectively. Then $\mathbb{M}''_{\hat{t}}$ can be expressed as:

$$\mathbb{M}''_{\hat{t}} = \mathbb{M}'_{\hat{t}} + (\beta''_{e',e} - \beta'_{e',e})(\Gamma_{e'} \cdot \mathbb{M}_e), \quad (4)$$

where $\Gamma_{e'}$ is the global encoding coefficient for edge e' (before the change) and $\mathbb{M}_e$ is $1 \times h$ matrix that ties the packet sent on edge e and the packets received by the incoming edges of $\hat{t}$. Note that the rank of $\Gamma_{e'} \times \mathbb{M}_e$ is at most one. The subadditivity property of the rank function implies that the rank of $\mathbb{M}''_{\hat{t}}$ is bounded by the rank of $\mathbb{M}'_{\hat{t}}$ plus one. ■

We proceed to establish an upper bound.

Lemma 2: Let $\hat{\mathbb{P}}$ be a set of h disjoint paths between s and $\hat{t}$ in $\hat{G}(\hat{V},\hat{E})$. Let $\hat{\mathbb{S}}$ be a set that contains pairs of adjacent edges in $\mathbb{S}$ such that each pair $(e',e) \in \hat{\mathbb{S}}$ belong to one of the paths in $\hat{\mathbb{P}}$. Then, a feasible network code can be constructed by changing the local encoding coefficients of the edges that belong to $\hat{\mathbb{S}}$.

Note that the lemma implies that it is sufficient to only change $|\hat{\mathbb{S}}|$ encoding coefficients.

Proof: First, we define a new network code $\hat{\mathbb{C}}$ as follows:

$$\hat{\beta}_{e',e} = \begin{cases} \beta_{e',e} + \Delta\beta_{e',e} & \text{if } (e',e) \in \mathbb{S} \text{ and } (e',e) \in \hat{\mathbb{P}} \\ \beta_{e',e} & \text{if } (e',e) \in \mathbb{S} \text{ and } (e',e) \notin \hat{\mathbb{P}} \\ \Delta\beta_{e',e} & otherwise. \end{cases} \quad (5)$$

We show that it is possible to assign the values of $\{\Delta\beta_{e',e}\}$ such that the resulting network code $\hat{\mathbb{C}}$ is feasible. We show that for each $t \in \hat{T}$ the determinant $\det(\hat{\mathbb{M}}_t)$ of the transfer matrix $\hat{\mathbb{M}}_t$ is not identically equal to zero with respect to the new network code $\hat{\mathbb{C}}$. To this end, we substitute the values of coefficients in $\{\beta_{e',e}\}$ according to their assignment in $\mathbb{C}$ and leave $\{\Delta\beta_{e',e}\}$ to be variables. Then, for each $t \in \hat{T}$ the determinant of the transfer matrix $\hat{\mathbb{M}}_t$ is a multivariate polynomial in $\{\Delta\beta_{e',e}\}$. We observe that for each $t \in T$ this polynomial is not identically equal to zero. Indeed, with the assignment of $\Delta\beta_{e',e} = 0$ for each pair of adjacent edges (e',e) in $\hat{\mathbb{P}}$ the transfer matrix $\hat{\mathbb{M}}_t$ for each $t \in T$ is identical to the transfer matrix $\mathbb{M}_t$ for the same terminal under code $\mathbb{C}$. For terminal $\hat{t}$ the multivariate polynomial $\det(\mathbb{M}_{\hat{t}})$ will include an additive term $\prod_{(e',e)\in\hat{\mathbb{P}}} \Delta\beta_{e',e}$, hence this polynomial is also not identically equal to zero. Therefore, for a sufficiently large field ($q \geq |\hat{T}|$) it is possible to select the values of $\{\Delta\beta_{e',e}\}$ such that the transfer matrix for each terminal are invertible. ■

The modified network code can be constructed through a simple modification of the algorithm due to Jaggi et. al. [5]. In addition, a random algorithm for the network code assignment can be used.

Figure 3 shows an instance of the dynamic network coding problem for which the bound established by Lemma 2 is tight. The initial network includes nodes $s, v_1, \ldots, v_9$. The existing network code $\mathbb{C}$ is shown in Figure 3(a). While the presented coding network is not minimal, all redundant edges can be justified by adding additional terminals. When a new terminal

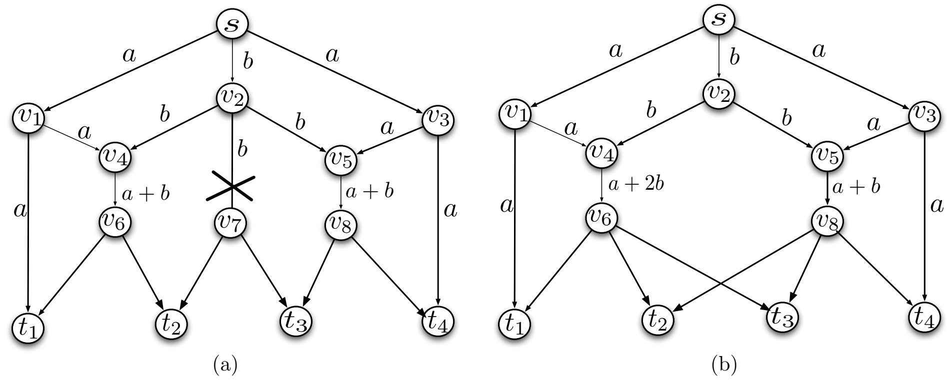

Fig. 2. (a) Original network $\mathbb{N}$ and a feasible network code $\mathbb{C}$ for $\mathbb{N}$. (b) A new network $\hat{\mathbb{N}}$ constructed after a failure of edge (v_2, v_7) with a new network code $\hat{\mathbb{C}}$ for $\hat{\mathbb{N}}$.

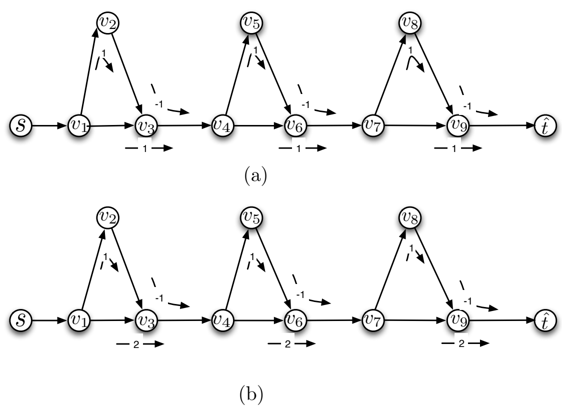

Fig. 3. An instance of a coding network. The arcs show local encoding coefficients between adjacent edges. (a) Original network code. (b) Modified network code.

$\hat{t}$ joins the network, it needs to receive at least one linear combination over path $\{s, v_1, v_3, v_4, v_6, v_7, v_9, \hat{t}\}$ that includes three pairs of adjacent edges. Figure 3(b) shows a modified network code that requires at least three changes.

V. Path-based assignment of encoding coefficients

In this section we present a path-based approach for the assignment of the local encoding coefficients. We first present our approach in the context of static networks and then discuss its operation in dynamic networks.

Let $\mathbb{N}$ be the original multicast network that needs to deliver h packets per communication round from source node s to a set of terminal nodes $t_i \in T$ over communication graph $G(V, E)$. The first step of our algorithm is to determine, for each terminal $t_i \in T$, a set $\mathcal{F}_i = \{P_1^i, \ldots, P_h^i\}$ of h edge-disjoint paths between s and t_i. Then, we associate each terminal t_i with the encoding parameter φ_i. For every two pairs of adjacent edges $e' = (u, v)$, $e = (v, w)$ we define a subset $T_{(e',e)}$ of T which includes all terminals $t_i \in T$ for which it holds that both $e = (u, v)$ and $e' = (v, w)$ belong to the same path in $\mathcal{F}_i$. Then, the local encoding coefficient $\beta_{e',e}$ is defined as follows:

$$\beta_{e,e'} = \sum_{t_i : t_i \in T_{(e,e')}} \varphi_i. \tag{6}$$

That is, the local encoding coefficient $\beta_{e',e}$ is defined to be the sum of the encoding parameters φ_i that correspond to the terminals in $T_{(e',e)}$.

We demonstrate our approach through the following example. Consider the communication network presented in Figure 4(a). The network includes a source node s and a set of terminals $T = \{t_1, t_2, t_3\}$. First, we identify three disjoint paths to each terminal in T (see Figures 4(b)-(d)). Then, we associate each terminal t_i with an encoding parameter φ_i. Then, for each pair of adjacent edges e' and e we assign the corresponding local encoding coefficient $\beta_{e',e}$ to be the sum of the coefficients that correspond to the terminals which include both edges (v, u) and (u, w) on one of their disjoint paths from the source. Figure 4(e) shows the local encoding coefficients assigned by our scheme for the network depicted in Figure 4(a). Our goal is to select the values of φ_i that yield a feasible network code. For the network depicted in Figure 4(a) it is easy to verify that $\varphi_1 = \varphi_2 = \varphi_3 = 1$ yields a feasible solution over $\mathbb{F}_3$.

Using the arguments similar to that used in Lemma 2, we can show for a sufficiently large field size $\mathbb{F}_q$ there always exists a feasible assignment of the encoding coefficients. Moreover, when a new terminal $\hat{t}$ joins the network, it is possible to find a feasible value of $\varphi_{\hat{t}}$ provided that the total number of active terminals is bounded. However, finding a feasible assignments of $\varphi_{\hat{t}}$ requires full knowledge of the network topology. In what follows we present an assignment of the encoding coefficients that does not requires full knowledge.

VI. Codes Based on Prime Numbers

Let $\mathcal{F}_i = \{P_1^i, \ldots, P_h^i\}$ be the set of h edge-disjoint paths between s and t_i determined in the first phase of the algorithm.

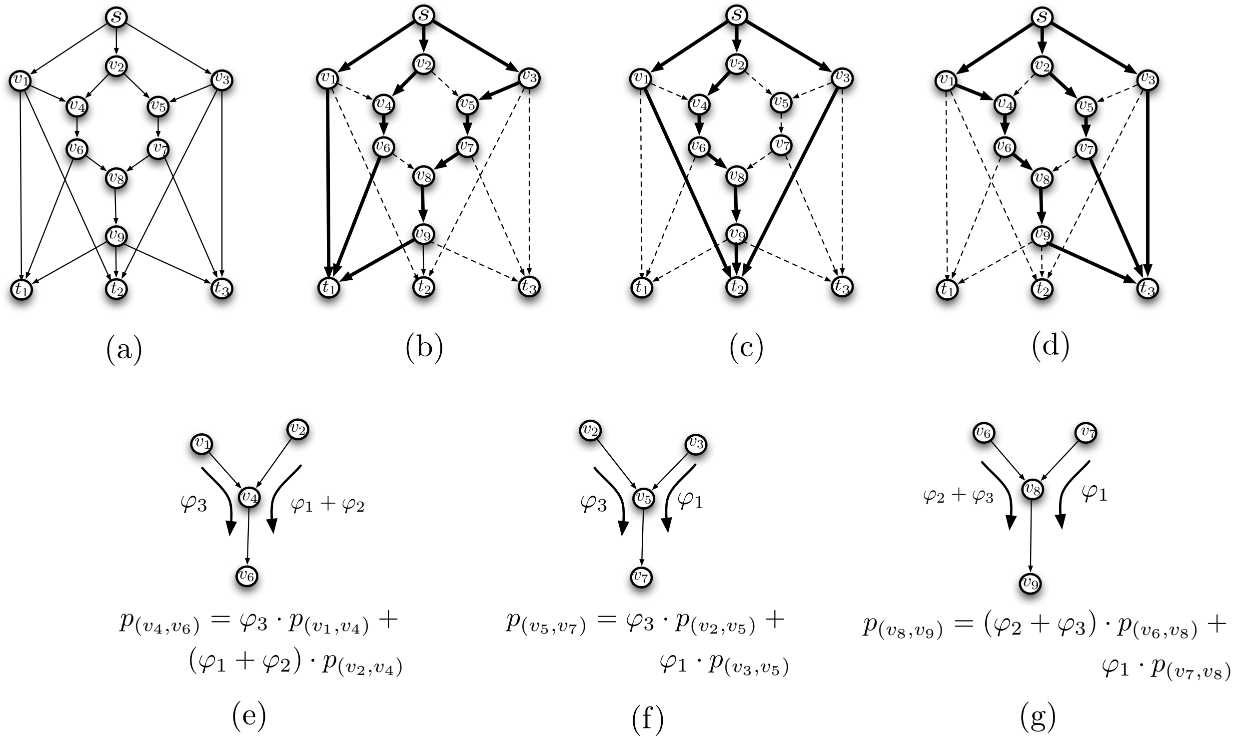

Fig. 4. (a) An instance to the network coding problem with three terminals t_1, t_2, and t_3; (b) Three disjoint paths to terminal t_1; (c) Three disjoint paths to terminal t_2; (d) Three disjoint paths to terminal t_3; (e) Local encoding coefficients for node v_4, v_5 and v_8. For example the packet $p_{(v_4,v_6)}$ sent on edge (v_4, v_6) is equal to $p_{(v_4,v_6)} = \varphi_3 \cdot p_{(v_1,v_4)} + (\varphi_1 + \varphi_2) \cdot p_{(v_2,v_4)}$.

Let also θ_j denote the j'th prime number (i.e., $\theta_1 = 2, \theta_2 = 3, \theta_3 = 5, \dots$). Let π_k be the product of the first k numbers, i.e.,

$$\pi_k = \prod_{j=1}^{k} \theta_j.$$

Define the encoding parameter φ_i for terminal t_i as follows:

$$\varphi_i = \pi_k/\theta_i = \prod_{j=1, j\neq i}^{k} \theta_j. \tag{7}$$

Figure 5 shows an example of network code based on prime numbers for a network with two terminals.

In Theorem 3 below we show that the assignment of the local encoding coefficients given by Equation (7) is feasible over a sufficiently large primary field.

Theorem 3: Let $\mathbb{F}_q$ be a primary field such that

$$q > (2 \cdot \pi_k)^{|E|},$$

where $|E|$ is the number of edges in a network. Then, the path-based assignment of the local encoding coefficients over $\mathbb{F}_q$ given by Equation (7) results in a feasible network code.

Proof: Let $\mathbb{N}(G(V,E), s, T)$ be a coding network and let $\{\beta_{e_j,e_i}\}$ be a set of local encoding coefficients for the adjacent edges in $\mathbb{N}$. Let $\mathcal{F}$ denote a flow composed of s h edge-disjoint paths $\{P_1, \dots, P_h\}$ between s and a terminal in T. We say that a pair (e_i, e_j) of adjacent edges belongs to $\mathcal{F}$ if there exists a path P_i in $\mathcal{F}$ such that both edges, e_i and e_j belong to P_i. We denote by

$$g(\mathcal{F}) = \prod_{\substack{(e_i,e_j) \text{ adjacent,} \\ (e_i,e_j)\in\mathcal{F}}} \beta_{e_i,e_j}$$

the *gain* of $\mathcal{F}$. Then, based on the result in [12, Th. 3], the network code $\{\beta_{e_j,e_i}\}$ is feasible, if and only if, for each terminal $t_i \in T$ it holds that

$$B_i = \sum_{\mathcal{F}:\mathcal{F} \text{ is a flow from } s \text{ to } t_i} \ell_i(\mathcal{F}) \cdot g(\mathcal{F}) \neq 0, \tag{8}$$

where $\ell_i(\mathcal{F}) \in \{1, -1\}$, and the summation is done over all flows between s and t_i.

Next, we will show that the choice of the φ_i's as described in Eq. (7) will lead to $B_i \neq 0$, for all $t_i \in T$. For every flow $\mathcal{F}$ from s to the terminal node t_i with a non-zero gain, one of the two following conditions holds:

1) Flow $\mathcal{F}$ is equal to flow $\mathcal{F}_i$. In this case $g(\mathcal{F})$ will include an additive term $\varphi_i^{N'} = (\pi_k/\theta_i)^{N'}$ for some integer N'. It easy to verify that all other multiple terms in $g(\mathcal{F})$ will have multiplicative factor θ_i.
2) Flows $\mathcal{F}$ and $\mathcal{F}_i$ are not identical. In this case, there exist adjacent edges $e' = (u,v)$, $e = (v,w)$ such that

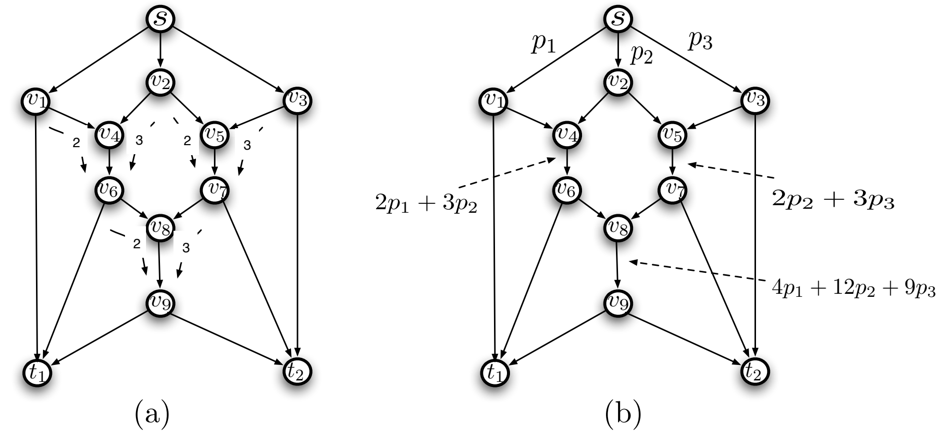

Fig. 5. (a) Local encoding coefficients, $\varphi_1 = \theta_2 = 3$ and $\varphi_2 = \theta_1 = 2$. (b) Global encoding coefficients.

both e' and e belong to a path in $\mathcal{F}$, but there is no path in $\mathcal{F}_i$ that includes both e' and e. In this case, every additive term in $g(\mathcal{F})$ is divisible by θ_i.

We conclude that B_i will include one additive term $(\pi_k/\theta_i)^{N'}$ for some integer N', while rest of the additive terms will include θ_i as a multiplicative factor. The Fundamental Theorem of Arithmetic implies that B_i is not equal to zero when addition is considered over the integers. We note that this also holds if the size q of the finite field is greater than B_i.

Next, we show that $(k \cdot \pi_k)^{|E|}$ is an upper bound on B_i. This will imply that for any prime field $\mathbb{F}_q$ such that $q \geq (2 \cdot \pi_k)^{|E|}$, B_i has a non-zero value.

First, we observe that each encoding coefficient is bounded by π_k. Second, any flow $\mathcal{F}$ from s to t contains at most $|E|$ edges, so the maximal gain $g(\mathcal{F})$ of $\mathcal{F}$ is at most $(\pi_k)^{|E|}$. We also note that the number of (s,t) flows is bounded by $2^{|E|}$. Thus, by Equation 8, $(2\pi_k)^{|E|}$ is an upper bound on the value of B_i and the theorem follows. ■

Our algorithms requires a finite field of size $(2\pi_k)^{|E|}$. It can be shown that the product of k prime numbers is bounded by 4^k. Then, the required size of the field is equal to $2^{(2k+1)|E|}$. Each element in such field can be represented by $(2k+1) \cdot |E|$ bits. Hence, the required packet size is linear in the number of edges in the network and the number of terminals.

VII. Conclusion and Future Work

This paper focuses on the design of efficient network coding algorithms for dynamic network settings. First, we establish upper and lower bounds on the number of required changes. Second, we present a path-based algorithm for assigning local network coding coefficients. The algorithm efficiently handles dynamic changes in the multicast group and the network topology.

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