1 Random sequence

Definition 1. An infinite sequence $X_n$, $n = 1, 2, \ldots$, of random variables is called a random sequence.

2 Convergence of a random sequence

Example 1. Consider the sequence of real numbers $X_n = \frac{n}{n+1}$, $n = 0, 1, 2, \ldots$

This sequence converges to the limit $l = 1$. We write

$$\lim_{n \to \infty} X_n = l = 1.$$

This means that in any neighbourhood around 1 we can trap the sequence, i.e.,

$$\forall \epsilon > 0, \exists n_0(\epsilon) \text{ s.t. } \text{for } n \geq n_0(\epsilon) \quad |X_n - l| \leq \epsilon.$$

We can pick $\epsilon$ to be very small and make sure that the sequence will be trapped after reaching $n_0(\epsilon)$. Therefore as $\epsilon$ decreases $n_0(\epsilon)$ will increase. For example, in the considered sequence:

$$\epsilon = \frac{1}{2}, \quad n_0(\epsilon) = 2,$$

$$\epsilon = \frac{1}{1000}, \quad n_0(\epsilon) = 1001.$$

2.1 Almost sure convergence

Definition 2. A random sequence $X_n$, $n = 0, 1, 2, 3, \ldots$, converges almost surely, or with probability one, to the random variable $X$ iff

$$P(\lim_{n \to \infty} X_n = X) = 1.$$

We write

$$X_n \xrightarrow{a.s.} X.$$
Example 2. Let $\omega$ be a random variable that is uniformly distributed on $[0, 1]$. Define the random sequence $X_n$ as $X_n = \omega^n$.

So $X_0 = 1, X_1 = \omega, X_2 = \omega^2, X_3 = \omega^3, \ldots$

Let us take specific values of $\omega$. For instance, if $\omega = \frac{1}{2}$

$X_0 = 1, X_1 = \frac{1}{2}, X_2 = \frac{1}{4}, X_3 = \frac{1}{8}, \ldots$

We can think of it as an urn containing sequences, and at each time we draw a value of $\omega$, we get a sequence of fixed numbers. In the example of tossing a coin, the output will be either heads or tails. Whereas, in this case the output of the experiment is a random sequence, i.e., each outcome is a sequence of infinite numbers.

Question: Does this sequence of random variables converge?

Answer: This sequence converges to

$$X = \begin{cases} 
0 & \text{if } \omega \neq 1 \text{ with probability } 1 = P(\omega \neq 1) \\
1 & \text{if } \omega = 1 \text{ with probability } 0 = P(\omega = 1) 
\end{cases}$$

Since the pdf is continuous, the probability $P(\omega = a) = 0$ for any constant $a$. Notice that the convergence of the sequence to 1 is possible but happens with probability 0.

Therefore, we say that $X_n$ converges almost surely to 0, i.e., $X_n \xrightarrow{a.s.} 0$.

2.2 Convergence in probability

Definition 3. A random sequence $X_n$ converges to the random variable $X$ in probability if

$$\forall \epsilon > 0 \lim_{n \to \infty} \Pr \{|X_n - X| \geq \epsilon\} = 0.$$  

We write:

$$X_n \xrightarrow{P} X.$$  

Example 3. Consider a random variable $\omega$ uniformly distributed on $[0, 1]$ and the sequence $X_n$ defined by:

$$X_n = \begin{cases} 
0 & \text{with probability } \frac{\omega}{n} \\
1 & \text{with probability } 1 - \frac{\omega}{n} 
\end{cases}$$

Distributed as shown in Figure [3]. Notice that only $X_2$ or $X_3$ can be equal to 1 for the same value of $\omega$. Similarly, only one of $X_4, X_5, X_6$ and $X_7$ can be equal to 1 for the same value of $\omega$ and so on and so forth.

Question: Does this sequence converge?
Answer: Intuitively, the sequence will converge to 0. Let us take some examples to see how the sequence behave.

for $\omega = 0$ : $1 \frac{10}{n=1} \frac{1000}{n=2} \frac{1000000}{n=3} \ldots$

for $\omega = \frac{1}{3}$ : $1 \frac{10}{n=1} \frac{0100}{n=2} \frac{00100}{n=3} \frac{001000}{n=4} \ldots$

From a calculus point of view, these sequences never converge to zero because there is always a “jump” showing up no matter how many zeros are preceding (Fig. 2); for any $\omega : X_n(\omega)$ does not converge in the “calculus” sense. Which means also that $X_n$ does not converge to zero almost surely (a.s.).
This sequence converges in probability since
\[
\lim_{n \to \infty} P (|X_n - 0| \geq 0) = 0 \quad \forall \epsilon > 0.
\]

Remark 1. The observed sequence may not converge in “calculus” sense because of the intermittent “jumps”; however the frequency of those “jumps” goes to zero when \( n \) goes to infinity.

2.3 Convergence in mean square

Definition 4. A random sequence \( X_n \) converges to a random variable \( X \) in mean square sense if
\[
\lim_{n \to \infty} E \left[ |X - X_n|^2 \right] = 0.
\]

We write:
\[
X_n \xrightarrow{m.s.} X.
\]

Remark 2. In mean square convergence, not only the frequency of the “jumps” goes to zero when \( n \) goes to infinity; but also the “energy” in the jump should go to zero.

Example 4. Consider a random variable \( \omega \) uniformly distributed over \([0, 1]\), and the sequence \( X_n(\omega) \) defined as:
\[
X_n(\omega) = \begin{cases} a_n & \text{for } \omega \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}
\]

Note that \( P (X_n = a_n) = \frac{1}{n} \) and \( P (X_n = 0) = 1 - \frac{1}{n} \).

Question: Does this sequence converge?
Answer: Let us check the different convergence criteria we have see so far.

1. **Almost sure convergence:** $X_n \overset{a.s.}{\longrightarrow} 0$ because
   \[
   \lim_{n \to \infty} P(X_n = 0) = 1.
   \]

2. **Convergence in probability:** $X_n \overset{p.}{\longrightarrow} 0$ because
   \[
   \lim_{n \to \infty} P(|X_n - 0| \leq \epsilon) = 0.
   \]
   (Flash Forward: almost sure convergence $\Rightarrow$ convergence in probability.)
   
   \[X_n \overset{a.s.}{\longrightarrow} X \Rightarrow X_n \overset{p.}{\longrightarrow} X.\]

3. **Mean Square Convergence:**
   \[
   E\left[|X_n - 0|^2\right] = a_n^2 (P(X_n = a_n) + 0P(X_n = 0)),
   \]
   \[
   = \frac{a_n^2}{n}.
   \]
   If $a_n = 10 \Rightarrow \lim_{n \to \infty} E\left[|X_n - 0|^2\right] = 0 \Rightarrow X_n \overset{m.s.}{\longrightarrow} 0,$
   If $a_n = \sqrt{n} \Rightarrow \lim_{n \to \infty} E\left[|X_n - 0|^2\right] = 1 \Rightarrow X_n$ does not converge in m.s. to 0.

In this example, the convergence of $X_n$ in the mean square sense depends on the value of $a_n$.

### 2.4 Convergence in distribution

**Definition 5.** *(First attempt)* A random sequence $X_n$ converges to $X$ in distribution if when $n$ goes to infinity, the values of the sequence are distributed according to a known distribution. We say

\[X_n \overset{d}{\longrightarrow} X.\]

**Example 5.** Consider the sequence $X_n$ defined as:

\[X_n = \begin{cases} 
X_i \sim B\left(\frac{1}{2}\right) & \text{for } i = 1 \\
(X_{i-1} + 1) \mod 2 = X \oplus 1 & \text{for } i > 1 
\end{cases}\]
Question: In which sense, if any, does this sequence converge?

Answer: This sequence has two outcomes depending on the value of $X_1$:

\[ X_1 = 1, \quad X_n : 1010101010 \ldots \]
\[ X_1 = 0, \quad X_n : 0101010101 \ldots \]

1. *Almost sure convergence:* $X_n$ does not converge almost surely because the probability of every jump is always equal to $\frac{1}{2}$.

2. *Convergence in probability:* $X_n$ does not converge in probability because the frequency of the jumps is constant equal to $\frac{1}{2}$.

3. *Convergence in mean square:* $X_n$ does not converge to $\frac{1}{2}$ in mean square sense because

\[
\lim_{n \to \infty} E\left[|X_n - \frac{1}{2}|^2\right] = E\left[X_n^2 - X_n + \frac{1}{4}\right],
\]
\[
= E[X_n^2] - E[X_n] + \frac{1}{4},
\]
\[
= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} - 0 + \frac{1}{4},
\]
\[
= \frac{1}{2}.
\]

4. *Convergence in distribution:* At infinity, since we do not know the value of $X_1$, each value of $X_n$ can be either 0 or 1 with probability $\frac{1}{2}$. Hence, any number $X_n$ is a random variable $\sim B(\frac{1}{2})$. We say, $X_n$ converges in distribution to Bernoulli(\(\frac{1}{2}\)) and we denote it by:

\[ X_n \overset{d}{\to} Ber(\frac{1}{2}). \]

**Example 6. (Central Limit Theorem)** Consider the zero-mean, unit-variance, independent random variables $X_1, X_2, \ldots, X_n$ and define the sequence $S_n$ as follows:

\[ S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}. \]

The CLT states that $S_n$ converges in distribution to $N(0, 1)$, i.e.,

\[ S_n \overset{d}{\to} N(0, 1). \]

**Theorem 1.**

\[
\text{Almost sure convergence, Convergence in mean square} \quad \Rightarrow \quad \text{Convergence in probability} \quad \Rightarrow \quad \text{convergence in distribution.}
\]

**Note:**

- There is no relation between Almost Sure and Mean Square Convergence.
- The relation is unidirectional, i.e., convergence in distribution does not imply convergence in probability neither almost sure convergence nor mean square convergence.
3 Convergence of a random sequence

Example 1: Let the random variable U be uniformly distributed on [0, 1]. Consider the sequence defined as:

$$X(n) = \frac{(-1)^n U}{n}.$$ 

Question: Does this sequence converge? if yes, in what sense(s)?

Answer:

1. Almost sure convergence: Suppose

   $$U = a.$$ 

   The sequence becomes

   $$X_1 = -a,$$
   $$X_2 = \frac{a}{2},$$
   $$X_3 = -\frac{a}{3},$$
   $$X_4 = \frac{a}{4},$$
   $$\vdots$$

   In fact, for any $$a \in [0, 1]$$

   $$\lim_{n \to \infty} X_n = 0,$$

   therefore, $$X_n \xrightarrow{a.s.} 0.$$ 

   Remark 3. $$X_n \xrightarrow{a.s.} 0$$ because, by definition, a random sequence converges almost surely to the random variable X if the sequence of functions $$X_n$$ converges for all values of U except for a set of values that has a probability zero.

2. Convergence in probability: Does $$X_n \xrightarrow{p} 0$$? Recall from theorem 13 of lecture 17:

   $$\text{a.s.} \Rightarrow \text{p.} \Rightarrow \text{d.}$$

   which means that by proving almost-sure convergence, we get directly the convergence in probability and in distribution. However, for completeness we will formally prove that $$X_n$$ converges to 0 in probability. To do so, we have to prove that

   $$\lim_{n \to \infty} P(|X - 0| \geq \epsilon) = 0 \quad \forall \epsilon > 0,$$

   $$\Rightarrow \lim_{n \to \infty} P(|X_n| \geq \epsilon) = 0 \quad \forall \epsilon > 0.$$
By definition, 
\[ |X_n| = \frac{U}{n} \leq \frac{1}{n}. \]

Thus,
\[
\begin{align*}
\lim_{n \to \infty} P\left(|X_n| \geq \epsilon\right) &= \lim_{n \to \infty} P\left(\frac{U}{n} \geq \epsilon\right), \\
&= \lim_{n \to \infty} P\left(U \geq n\epsilon\right), \\
&= 0.
\end{align*}
\]

Where equation 3 follows from the fact that finding \( U \in [0, 1] \).

3. **Convergence in mean square sense:** Does \( X_n \) converge to 0 in the mean square sense?

In order to answer this question, we need to prove that
\[ \lim_{n \to \infty} E\left[|X_n - 0|^2\right] = 0. \]

We know that,
\[
\begin{align*}
\lim_{n \to \infty} E\left[|X_n - 0|^2\right] &= \lim_{n \to \infty} E\left[X_n^2\right], \\
&= \lim_{n \to \infty} E\left[\frac{U^2}{n^2}\right], \\
&= \lim_{n \to \infty} \frac{1}{n^2} E\left[U^2\right], \\
&= \lim_{n \to \infty} \frac{1}{n^2} \int_0^1 u^2 \, du, \\
&= \lim_{n \to \infty} \frac{1}{n^2} \left[ \frac{u^3}{3} \right]_0^1 \\
&= \lim_{n \to \infty} \frac{1}{3n^2}, \\
&= 0.
\end{align*}
\]

Hence, \( X_n \xrightarrow{m.s.} 0. \)

4. **Convergence in distribution:** Does \( X_n \) converge to 0 in distribution? The formal definition of convergence in distribution is the following:
\[ X_n \xrightarrow{d} X \Rightarrow \lim_{n \to \infty} F_{X_n}(x) = F_X(x). \]

Hereafter, we want to prove that \( X_n \xrightarrow{d} 0. \)

Recall that the limit r.v. \( X \) is the constant 0 and therefore has the following CDF:
Since \( X_n = \frac{(-1)^n U}{n} \), the distribution of the \( X_i \) can be derived as following:
Remark 4. At 0 the CDF of $X_n$ will be flip-flopping between 0 (if $n$ is even) and 1 (if $n$ is odd) (c.f. figure 3) which implies that there is a discontinuity at that point. Therefore, we say that $X_n$ converges in distribution to a CDF $F_X(x)$ except at points where $F_X(x)$ is not continuous.

Definition 6. $X_n$ converges to $X$ in distribution, i.e., $X[n] \xrightarrow{d} X$ iff
\[
\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \quad \text{except at points where } F_X(x) \text{ is not continuous.}
\]

Remark 5. It is clear here that
\[
\lim_{n \to \infty} F_{X_n}(x) = F_X(x) \quad \text{except for } x = 0.
\]

Therefore, $X_n$ converges to $X$ in distribution. We could have deduced this directly from convergence in mean square sense or almost sure convergence.

Theorem 2. a) If $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p} X$.

b) If $X_n \xrightarrow{m.s.} X \Rightarrow X_n \xrightarrow{p} X$.

c) If $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$.

d) If $P\{|X_n| \leq Y\} = 1$ for all $n$ for a random variable $Y$ with $E[Y^2] < \infty$, then
\[
X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{m.s.} X.
\]

Proof. The proof is omitted.

Remark 6. Convergence in probability allows the sequence, at $\infty$, to deviate from the mean for any value with a small probability; whereas, convergence in mean square limits the amplitude of this deviation when $n \to \infty$. (We can think of it as energy $\Rightarrow$ we can not allow a big deviation from the mean).
4 Back to real analysis

Definition 7. A sequence \((x_n)_{n \geq 1}\) is Cauchy if for every \(\epsilon\), there exists a large number \(N\) s.t.
\[
\forall m, n > N, \ |x_m - x_n| < \epsilon \iff \lim_{n,m \to \infty} |x_m - x_n| = 0.
\]

Claim 1. Every Cauchy sequence is convergent.

Counter example 1. Consider the sequence \(X_n \in \mathbb{Q}\) defined as \(x_0 = 1, x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}\). The limit of this sequence is given by:
\[
l = \frac{l + \frac{2}{l}}{2},
\]
\[
2l^2 = l^2 + 2,
\]
\[
l = \pm\sqrt{2} \notin \mathbb{Q}.
\]

This implies that the sequence does not converge in \(\mathbb{Q}\).
Counter example 2. Consider the sequence \( x_n = 1/n \) in \((0,1)\). Obviously it does not converge in \((0,1)\) since the limit \( l = 1 \notin (0,1) \).

**Definition 8.** A space where every sequence converges is called a complete space.

**Theorem 3.** \( \mathbb{R} \) is a complete space.

**Proof.** The proof is omitted. \( \square \)

**Theorem 4.** Cauchy criteria for convergence of a random sequence.

\[
\begin{align*}
a) & \quad X_n \xrightarrow{a.s.} X \iff P \left( \lim_{m,n \to \infty} |x_m - x_n| = 0 \right) = 1. \\
b) & \quad X_n \xrightarrow{m.s.} X \iff \lim_{m,n \to \infty} E \left[ |x_m - x_n|^2 \right] = 0. \\
c) & \quad X_n \xrightarrow{p.} X \iff \lim_{m,n \to \infty} P \left[ |x_m - x_n| \geq \varepsilon \right] = 0 \quad \forall \varepsilon.
\end{align*}
\]

**Proof.** The proofs are omitted. \( \square \)

**Example 7.** Consider the sequence of example 11 from last lecture,

\[
X_n = \begin{cases} 
X_i \sim B\left(\frac{1}{2}\right) & \text{for } i = 1 \\
(X_{i-1} + 1) \mod 2 = X \oplus 1 & \text{for } i > 1 
\end{cases}
\]

**Goal:** Our goal is to prove that this sequence does not converge in mean square using Cauchy criteria.

This sequence has two outcomes depending on the value of \( X_1 \):

\[
X_1 = 1, \quad X_n : 101010101010\ldots \\
X_1 = 0, \quad X_n : 010101010101\ldots 
\]

Therefore,

\[
E \left[ |X_n - X_m|^2 \right] = E \left[ X_n^2 \right] + E \left[ X_m^2 \right] - 2E \left[ X_m X_n \right],
\]

\[
= \frac{1}{2} + \frac{1}{2} - 2E \left[ X_m X_n \right].
\]

Consider, without loss of generality, that \( m > n \)

\[
E \left[ X_n X_m \right] = \begin{cases} 
E \left[ X_n X_m \right] = 0 & \text{if } m - n \text{ is odd,} \\
E \left[ X_n^2 \right] = \frac{1}{2} & \text{if } m - n \text{ is even.}
\end{cases}
\]

Hence,

\[
\lim_{n,m \to \infty} E \left[ |X_n - X_m|^2 \right] = \begin{cases} 
1 & \text{if } m - n \text{ is odd,} \\
0 & \text{if } m - n \text{ is even},
\end{cases}
\]

which implies that \( X_n \) does not converge in mean square by theorem \( \text{[4][b]} \)
**Lemma 1.** Let $X_n$ be a random sequence with $E[X_n^2] < \infty \ orall n$.

$$X_n \overset{m.s.}{\to} X \quad \text{iff} \quad \lim_{m,n \to \infty} E[X_mX_n] \text{ exists and is finite.}$$

**Theorem 5.** Weak law of large numbers

Let $X_1, X_2, X_3, \ldots, X_i$ be iid random variables. $E[X_i] = \mu$, $\forall i$. Let

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

Then

$$P[|S_n - \mu| \geq \epsilon] \to 0 \quad \text{as } n \to \infty.$$

Using the language of this chapter:

$$S_n \overset{p}{\to} \mu.$$

**Theorem 6.** Strong law of large numbers

Let $X_1, X_2, X_3, \ldots, X_i$ be iid random variables. $E[X_i] = \mu$, $\forall i$. Let

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

Then

$$P\left[\lim_{n \to \infty} |S_n - \mu| \geq \epsilon\right] = 0.$$

Using the language of this chapter:

$$S_n \overset{a.s.}{\to} \mu.$$

**Theorem 7.** Central limit theorem

Let $X_1, X_2, X_3, \ldots, X_i$ be iid random variables. $E[X_i] = 0$, $\forall i$. Let

$$Z_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$

Then

$$P[Z_n \leq z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

Using the language of this chapter:

$$Z_n \overset{d}{\to} N(0, 1).$$