ECE511: Analysis of Random Signals Fall 2016

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Chapter 1: Introduction to Probability Theory

1 Random Experiment

Definition 1 (Sample space). *The* Sample space*, Ω, is the set of all possible outcomes.*

Example 1. *When we toss a coin, all the possible outcomes are Heads or Tails. Therefore, the sample space of a coin toss is $\Omega = \{Heads,\ Tails\}$.*

Definition 2 (Event). *An event E is a subset of the sample space, i.e., $E \subseteq \Omega$.*

Example 2. *If we toss a fair coin twice, then the sample space is $\Omega = \{HH,\ HT,\ TH,\ TT\}$. Consider the event A "at least one Head occurs"; then, the event is $A = \{HH,\ HT,\ TH\}$.*

Let B be the event of tossing the coin repeatedly until a Head occurs. Then, $B = \{H,\ TH,\ TTH, \dots\}$. Let C be the event of tossing the coin an even number of times until a Head occurs. Then, $C = \{TH,\ TTTH, \dots\}$.

Remark 1. *Not all subset of Ω are events. You can define sets that have no probability. For this class, any subset of Ω is an event.*

Definition 3 (Axioms of probability). *A probability measure P on Ω is a function*

$$\begin{aligned} P : 2^{\Omega} &\to [0,\ 1], \\ E &\to P(E), \end{aligned}$$

such that it satisfies the following properties:

(1) $P(\emptyset) = 0$.

(2) $P(\Omega) = 1$.

(3) If $A_1,\ A_2,\ A_3 \dots$ are disjoint subsets of Ω,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_i P(A_i).$$

Here, 2^{Ω} is the power set of Ω.

Lemma 1. *Let A and B be two subsets of Ω. We define $\bar{A}$ to be the complement of A in Ω, we have:*

(a) $P(\bar{A}) = 1 - P(A)$.

(b) If $A \subseteq B$*, then* $P(A) \leq P(B)$.

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. For part(a),

$$\begin{aligned} & P(A \cup \bar{A}) = P(\Omega) = 1 \text{ and } A,\, \bar{A} \text{ are disjoint} \\ & \Rightarrow P(A) + P(\bar{A}) = 1 \\ & \Rightarrow P(\bar{A}) = 1 - p(A) \end{aligned}$$

For part(b),

$$\begin{aligned} B &= A \cup (B \setminus A) \\ \Rightarrow P(B) &= P(A) + P(B \setminus A) \\ &\geq P(A) \end{aligned}$$

For part(c),

$$P(A \cup B) = P(A) + P(B|A) = P(A) + P(B|A \cap B).$$

Now,

$$(A \cap B) \subseteq B \Rightarrow P(B|A \cap B) = P(B) - P(A \cap B).$$

□

Lemma 2 (Union bound)**.** *Let* A *and* B *be two subsets of* Ω*, then*

$$P(A \cup B) \leq P(A) + P(B).$$

In general,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

Example 3 (Tossing a Die (a))**.** A_1*: The result number is a multiple of 2.* A_2*: The result number is a multiple of 3.*

$A_1 = \{2, 4, 6\}$, $P(A_1) = \frac{1}{2}$. $A_2 = \{3, 6\}$, $P(A_2) = \frac{1}{3}$.

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

In fact, $A_1 \cup A_2 = \{2, 3, 4, 6\}$ *and* $P(A_1 \cup A_2) = \frac{2}{3}$.

Example 4 (Tossing a Die (b))**.** A_1*: The result number is less than or equal to 3.* A_2*: The result number is prime.*

$A_1 = \{3, 4, 5, 6\}$, $P(A_1) = \frac{2}{3}$. $A_2 = \{2, 3, 5\}$, $P(A_2) = \frac{1}{2}$.

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2) = \frac{2}{3} + \frac{1}{2} = \frac{7}{6} > 1$$

In fact, $A_1 \cup A_2 = \{2, 3, 4, 5, 6\}$ *and* $P(A_1 \cup A_2) = \frac{5}{6}$.

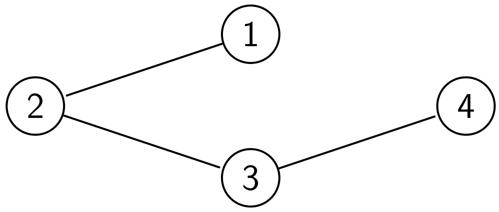
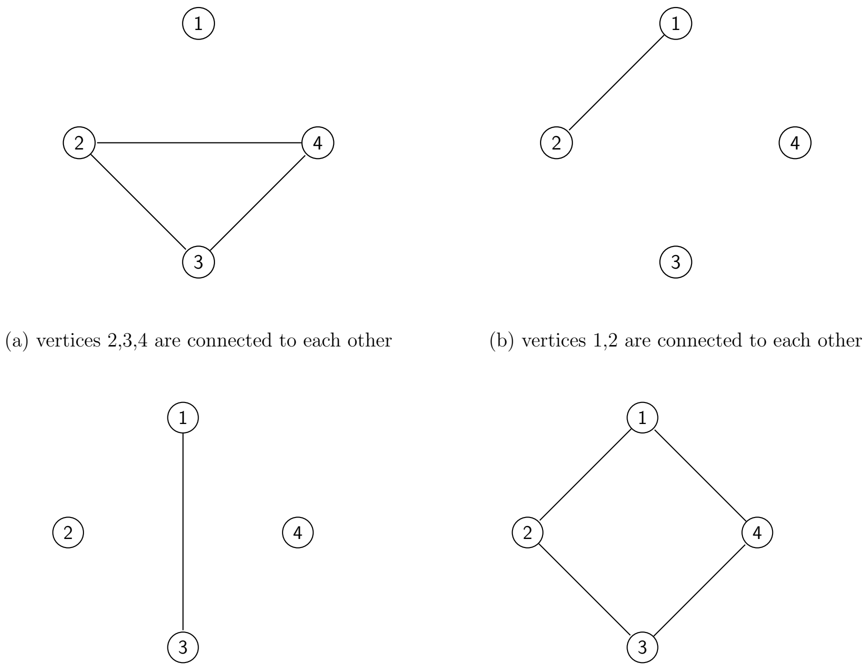


Figure 1: Graph connection.

Example 5 (Random Graphs)**.** *Consider the graph $\mathcal{G} = (V, E)$ over 4 vertices, given in Figure 1, where $V = \{1, 2, 3, 4\}$ is the vertex set and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ is the edge set.*

A random graph $\mathcal{G}$ defined over the vertex set V is a graph where an edge between any two vertices exists with a probability p. If we take a graph on n vertices and the edge exists between 2 vertices with probability=p=0.5. Then the number of subsets of V of size $2 = \frac{n(n-1)}{2} = \binom{n}{2}$. The number of subsets of V of size $k = \binom{n}{k}$.

If we have 4 vertices in a graph. What is the probability that vertex 1 is connected to k other nodes?



(a) vertices 2,3,4 are connected to each other

(b) vertices 1,2 are connected to each other

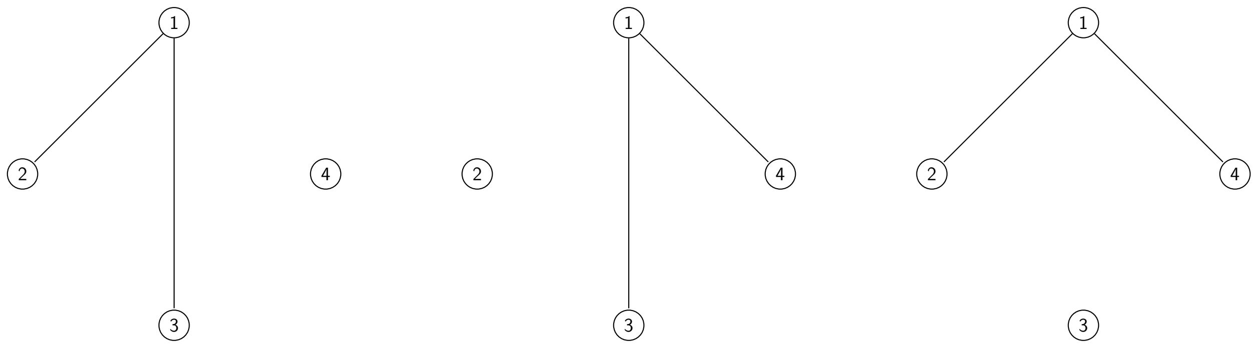
(c) vertices 1,3 are connected to each other

(d) vertices 1,2,3,4 connected to each other clockwise

Figure 2: Graph connection for 4 vertices

Let N be the neighbors of vertex 1, $N = \phi$ in fig(a), $N = (1, 2)$ in fig(b), $N = (1, 3)$ in fig(c), $N = (2, 3, 4)$ in fig(d).
Then we define the event A_N is that the vertex 1 is connected to the vertices in N

We say vertex 1 is connected to k other vertices, if k =2, all the possible graph are as (Figure 3).



(a): Node 1 connected to 2 and 3 (b): Node 1 connected to 3 and 4. (c): Node 1 connected to 2 and 4.

Figure 3: vertex 1 is connected to two vertices

Define event A vertex 1 is connected to 2 other vertices,therefore:

$$A = A_{\{2,3\}} \cup A_{\{3,4\}} \cup A_{\{2,4\}}.$$

The probability of this event A is

$$P(A) = P(A_{\{2,3\}}) + P(A_{\{3,4\}}) + P(A_{\{2,4\}}).$$

The probability of vertex 1 is connected to vertex 2 and 3 is

$$P(A_{\{2,3\}}) = (\frac{1}{2})^3 = p^2(1-p) = P(A_{\{3,4\}}) = P(A_{\{2,4\}}),$$

therefore,

$$P(A) = 3p^2(1-p).$$

In general, the probability vertex 1 *is connected to k specific vertices is*

$$P(A_N) = p^k(1-p)^{n-1-k}.$$

The probability vertex 1 *is connected to k other vertices is*

$$\begin{aligned} P(A) &= \Sigma P(A_N), \\ &= \binom{n-1}{k} p^k (1-p)^{n-1-k}. \end{aligned}$$

1.1 Conditional Probability

Example 6. *Let A be the event of tossing two fair dice such that the total exceeds 6.*

(a) Find $P(A)$. The set of all events Ω is given by the following set:

$$\Omega = \left\{ \begin{array}{cccc} (1,1), & (1,2), & \ldots & (1,4), \\ (2,1), & (2,2), & \ldots & (2,6), \\ \vdots & \vdots & \ddots & \vdots \\ (6,1), & (6,2), & \ldots & (6,6). \end{array} \right\}.$$

Now, we need to find $P(A)$. All the possible outcomes of A (total exceeds 6) are:

$$\begin{aligned} A = \{&(1,6)\\ &(2,5),\ (2,6)\\ &(3,4),\ (3,5),\ (3,6)\\ &(4,3),\ (4,4),\ (4,5),\ (4,6)\\ &(5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6)\\ &(6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6)\}. \end{aligned}$$

We know that

$$\begin{aligned} P(A) &= \frac{\textit{Number of possible outcomes}}{\textit{Number of all outcomes}}\\ P(A) &= \frac{21}{36}. \end{aligned}$$

(b) Let B the event that the first dice shows 3. Find $P(B)$.
All the possible outcomes of event B are:

$$B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}. \tag{1}$$

Then all the possible outcomes of event A given B are the events in equation (1) *satisfying A (total exceeds 6), hence*

$$(A \cap B) = \{(3,4),(3,5),(3,6)\}.$$

So the conditional probability of A given B is

$$P(A|B) = \frac{3}{6},$$

we can find that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 4 (Conditional probability)**.** *We define the conditional probability of an event A given that event B happened (with $P(B) > 0$) by:*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 5 (Independent events)**.** *Two events A and B are independent iff*

$$P(A \cap B) = P(A)P(B).$$

In general,

$$\begin{aligned} P(A \cap B) &= P(A)P(B|A) &(2)\\ &= P(B)P(A|B). &(3) \end{aligned}$$

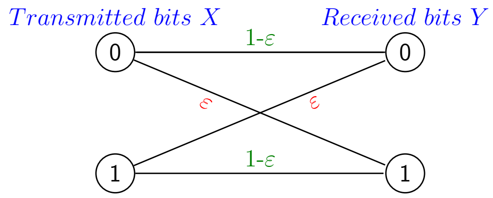


Figure 4: Binary Symmetric Channel (BSC) with probability of error $P_e = \varepsilon$.

We can also say that the events A and B are independent iff

$$\begin{aligned} P(A|B) &= P(A), \quad (P(B) \neq 0) \\ P(B|A) &= P(B), \quad (P(A) \neq 0). \end{aligned}$$

Example 7 (Binary symmetric channel)**.**
In the BSC of Fig. 8 the bits are flipped with probability ε (ε is called crossover probability), we can write

$$\begin{aligned} \varepsilon &= P(Y = 0|X = 1) \\ &= P(Y = 1|X = 0). \end{aligned}$$

Suppose the bits '0' and '1' are equal likely to be sent, i.e.,

$$P(X = 0) = P(X = 1) = 0.5,$$

Q. *Find the probability of sending a '0' and receiving a '0'.*

Ans.

$$\begin{aligned} P(X = 0, Y = 0) &= P(X = 0)P(Y = 0|X = 0) \\ &= 0.5(1 - \varepsilon). \end{aligned}$$

1.2 Total Law of Probability

Theorem 1. *Let $A_1, A_2, \ldots, A_n$ be n mutually disjoint events such that*

$$\Omega = \bigcup_{i=1}^{n} A_i \ (P(A_i) \neq 0), \tag{4}$$

then for any event $B \subseteq \Omega$ we have

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + ... + P(A_n)P(B|A_n).$$

Proof. For n=2

$$\begin{aligned} B &= (B \cap A_1) \cup (B \cap A_2), &(5)\\ P(B) &= P(B \cap A_1) + P(B \cap A_2), &(6)\\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2). &(7) \end{aligned}$$

□

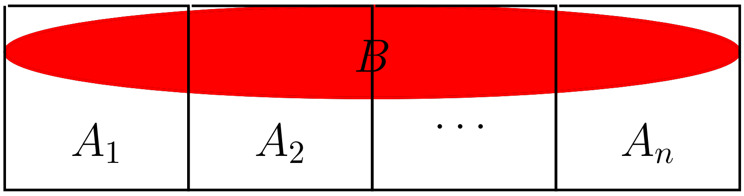


Figure 5: Total law of probability.

Example 8. *(BSC) Consider a BSC in Fig. 6 with crossover probability* $\varepsilon = 0.1$*. The probability of sending '0' is 0.4 and the probability of sending '1' is 0.6.*

Q. *Find the probability of receiving a '0'.*

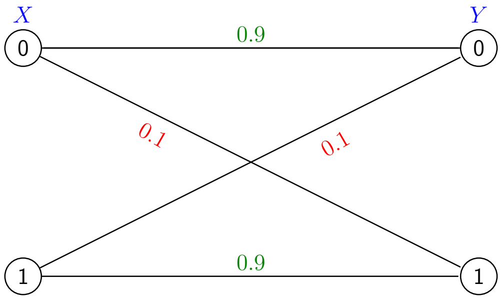


Figure 6: Binary Symmetric Channel with probability of error $P_e = 0.1$.

Ans. *The probability of sending '1' is* $P(X = 1) = 0.6$*, and the probability of sending '0' is* $P(X = 0) = 0.4$*. Then if we want to know the probability of receiving '0', we can use the total law of probability to calculate* $P(Y = 0)$*,*

$$\begin{aligned} P(Y=0) &= P(X=0)P(Y=0|X=0) + P(X=1)P(Y=0|X=1), \\ &= (0.4) \times (0.9) + (0.6) \times (0.1) = 0.42. \end{aligned}$$

1.3 Birthday paradox

Question: What is the probability that at least 2 students in class have the same birthday.
E:at least 2 students have the same birthday.
Number of birthdays per year is n, number of students in class is m.
$\bar{E}$: each student has distinct birthday.
Answer:

$$P(\bar{E}) = 1 \times (1 - \frac{1}{n}) \times (1 - \frac{2}{n}) \times \cdots \times (1 - \frac{m-1}{n}).$$

We know that

$$1 - \frac{k}{n} \approx \mathrm{e}^{-\frac{k}{n}}, \quad k \ll n.$$

Then,

$$\begin{aligned} P(\bar{E}) &= e^{-\frac{1}{n}} \times e^{-\frac{2}{n}} \times \cdots \times e^{-\frac{m-1}{n}}, \\ &= exp(-\frac{1}{n}(1+2+\cdots+m-1)), \\ &= e^{-\frac{m(m-1)}{2n}}, \\ &\approx e^{-\frac{m^2}{2n}}. \end{aligned}$$

Now we have student $m = 50$, and number of birthdays $n = 365$.

$$\begin{aligned} P(E) &\approx 1 - e^{-\frac{50^2}{2\times 365}}, \\ &\approx 96.7\%. \end{aligned}$$

Question: How big the class should be if the probability of 2 students have same birthday is larger than 50%?
Answer:

$$P(E) = \frac{1}{2}.$$

Then

$$1 - e^{-\frac{m^2}{2n}} = \frac{1}{2},$$

so

$$\frac{m^2}{2n} = \ln 2,$$

$$\begin{aligned} m &= \sqrt{2\ln 2} \times \sqrt{n}, \\ &\approx 23. \end{aligned}$$

So we need approximately 23 students in same class to make the probability that at least 2 students have the same birthday is larger than $\frac{1}{2}$.

Theorem 2 (Baye's Theorem)**.**

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}. \quad (8)$$

Example 9 (BSC)**.** *In this case we have* $P(X = 0) = P(X = 1) = \frac{1}{2}$ *(0s and 1s are equal likely transmitted)*
Suppose we observe $Y = 1$*. What value of* X *should we decode?*

$$P(X = 1|Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)}.$$

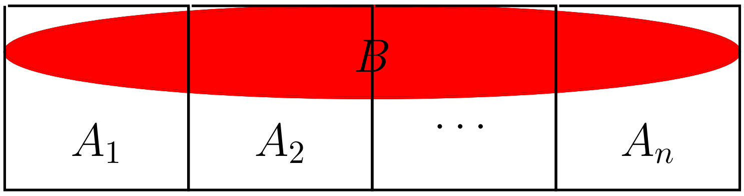


Figure 7: Baye's theorem.

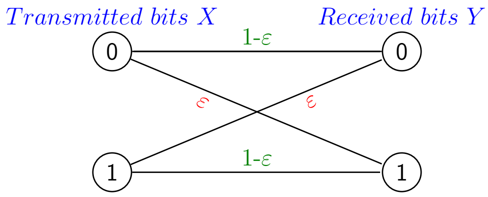


Figure 8: Binary Symmetric Channel (BSC) with probability of error $P_e = \epsilon$.

According to the Baye's theorem

$$\begin{aligned} P(X=1|Y=1) &= \frac{P(X=1)P(Y=1|X=1)}{P(X=0)P(Y=1|X=0)+P(X=1)P(Y=1|X=1)}, \\ &= \frac{0.5(1-\varepsilon)}{0.5\varepsilon+0.5(1-\varepsilon)}, \\ &= 1-\varepsilon. \end{aligned}$$