ECE541: Stochastic Signals and Systems Fall 2018

Chapter 8: Random Processes

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1 Random Process

Definition 1. *A discrete random process $X(n)$, also denoted as X_n, is an infinite sequence of random variables $X_1, X_2, X_3, \ldots$; we think of n as the time index.*

1. Mean function: $\mu_X(n) = E[X(n)]$.
2. Auto-correlation function: $R_{XX}(k,l) = E[X(k)X(l)]$.
3. Auto-covariance function: $K_{XX}(k,l) = R_{XX}(k,l) - \mu_X(k)\mu_X(l)$.

Definition 2. *$X(n)$ is a Gaussian r.p. if $X_{i_1}, X_{i_2}, \ldots, X_{i_m}$ are jointly Gaussian for any $m \geq 1$.*

Example 1. *Let $W(n)$ be an i.i.d Gaussian r.p with autocorrelation $R_{WW}(k,l) = \sigma^2\delta(k-l)$ and mean $\mu_W(n) = 0\,\forall n$, where*

$$\delta(x-a) = \begin{cases} 1 & x = a, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\Rightarrow R_{WW}(k,l) = E\left[W_k W_l\right] = \begin{cases} \sigma^2 & k = l, \\ 0 & \textit{otherwise.} \end{cases}$$

Which means that for $l = k$,

$$R_W(k,l) = E[W_l^2] = V[W_l] = \sigma^2,$$

and W_l & W_k are correlated. While for $l \neq k$,

$$R_W(k,l) = 0,$$

and W_l & W_k are uncorrelated and therefore independent because they are jointly Gaussian.

A new averaging r. p. $X(n)$ defined as

$$X(n) = \frac{W(n) + W(n-1)}{2} \quad n \geq 1,$$

for example for $n = 1$,

$$X(1) = \frac{W_1 + W_0}{2}.$$

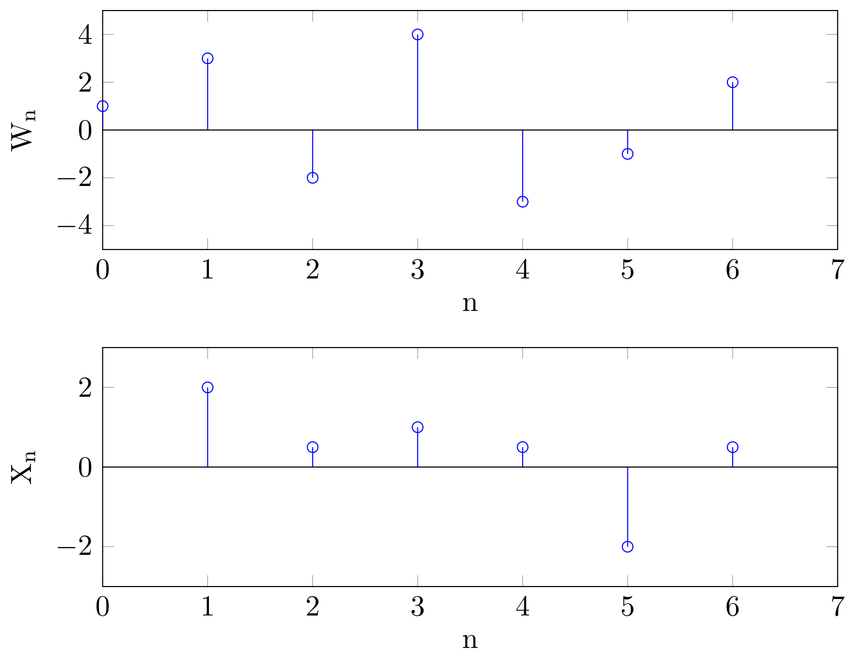

Figure 1: A possible realization of the random process $W(n)$ and its corresponding averaging function$X(n)$.

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Questions:

1. What is the pdf of $X(n)$?
2. Find the autocorrelation function $R_{XX}(k,l)$

Answers:

1. From previous chapters we know that $X(n)$ is a Gaussian r.v. because it is a linear combination of Gaussian r.v. Hence, it is enough to find the mean and variance of X_n to find its pdf.

$$
\begin{aligned}
E[X(n)] &= E\left[\frac{1}{2}(W_n + W_{n-1})\right], \\
&= \frac{1}{2}(E[W_n] + E[W_{n-1}]), \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
V[X(n)] &= V\left[\frac{1}{2}(W_n + W_{n-1})\right], \\
&= \frac{1}{4}(V[W_n] + V[W_{n-1}]) \quad \text{because } W_n \ \& \ W_{n-1} \text{ are independent}, \\
&= \frac{1}{4}\left(\sigma^2 + \sigma^2\right), \\
&= \frac{\sigma^2}{2}.
\end{aligned}
$$

OR:

$$
\begin{aligned}
V\left[X\left(n\right)\right] &= E\left[X_n^2\right] - \mu_{X_n}^2, && (1)\\
&= E\left[\frac{1}{4}\left(W_n + W_{n-1}\right)^2\right], && (2)\\
&= \frac{1}{4}\left(E\left[W_n^2\right] + E\left[W_{n-1}^2\right] + 2E\left[W_n W_{n-1}\right]\right), && (3)\\
&= \frac{1}{4}\left(\sigma^2 + \sigma^2\right), && (4)\\
&= \frac{\sigma^2}{2}, && (5)
\end{aligned}
$$

where equation (4) follows from the fact that

$$E\left[W_n W_{n-1}\right] = R_{WW}(n, n-1) = 0.$$

Therefore,

$$
\begin{aligned}
f_{X_n}(x_n) &= \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{2}}} \exp\left(-\frac{x_n^2}{2\frac{\sigma^2}{2}}\right),\\
&= \frac{1}{\sigma\sqrt{\pi}} \exp\left(-\frac{x_n^2}{\sigma^2}\right).
\end{aligned}
$$

2. Before we apply the formula, let us try to find the autocorrelation intuitively. By definition:

$$X_1 = \frac{W_1 + W_0}{2}, \quad X_2 = \frac{W_2 + W_1}{2}, \quad X_3 = \frac{W_3 + W_2}{2}.$$

It is clear that X_1 and X_3 are uncorrelated (independent) because they do not have any W_i in common and $W(n)$ is i.i.d. However, X_1, X_2 and X_2, X_3 are correlated.

$$
\begin{aligned}
R_{XX}(k,l) &= E\left[X_k X_l\right],\\
&= \frac{1}{4} E\left[\left(W_k + W_{k-1}\right)\left(W_l + W_{l-1}\right)\right],\\
&= \frac{1}{4}\left(E\left[W_k W_l\right] + E\left[W_k W_{l-1}\right] + E\left[W_{k-1} W_l\right] + E\left[W_{k-1} W_{l-1}\right]\right).
\end{aligned}
$$

Recall from the definition of $W(n)$ that

$$
E\left[W_k W_l\right] = \begin{cases} \sigma^2/2 & k = l,\\ 0 & \text{otherwise,} \end{cases} \qquad E\left[W_k W_{l-1}\right] = \begin{cases} \sigma^2/4 & k = l-1,\\ 0 & \text{otherwise,} \end{cases}
$$

$$
E\left[W_{k-1} W_l\right] = \begin{cases} \sigma^2/4 & k = l+1,\\ 0 & \text{otherwise,} \end{cases} \qquad E\left[W_{k-1} W_{l-1}\right] = \begin{cases} \sigma^2/2 & k = l,\\ 0 & \text{otherwise.} \end{cases}
$$

Therefore,

$$
R_{XX}(k,l) = \begin{cases} \frac{\sigma^2}{2} & k = l,\\ \frac{\sigma^2}{4} & k = l \pm 1,\\ 0 & \text{otherwise.} \end{cases}
$$

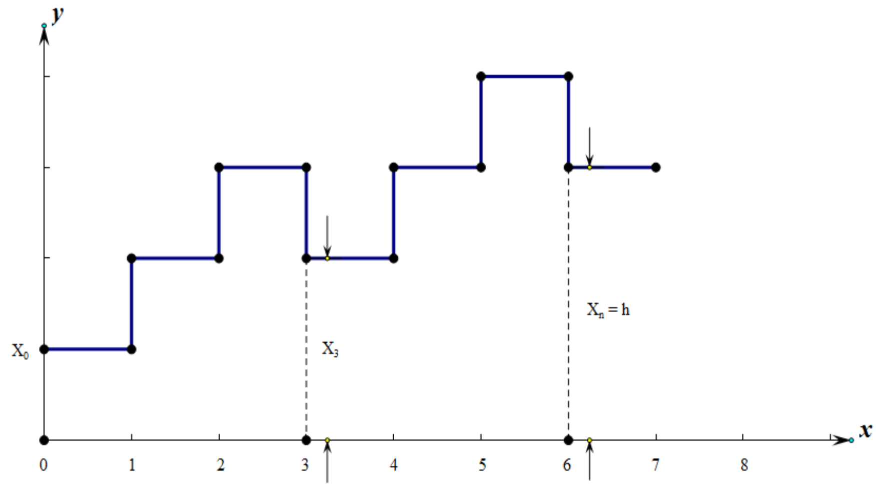


Figure 2: A possible realization of the random walk process

Example 2. *Random walk process*

Let $W_0 = X_0 = constant$, and $W_1, W_2, \ldots$ be i.i.d. random process with the following distribution

$$W_i = \begin{cases} 1 & p, \\ -1 & 1-p. \end{cases}$$

The random walk process X_n, $n = 1, 2, \ldots$ is then defined as

$$X_n = W_0 + W_1 + W_2 + \cdots + W_n.$$

Questions:

1. What is the pdf of X_n?
2. Find the mean function of X_n.
3. Find the variance of X_n.

Answers:

1. X_n is a Binomial r.v. since it is the summation of Bernoulli r.v. So to find the pdf of X_n is the same as finding $P(X_n = h)$. Let U be the number of steps "up", i.e., the corresponding $W_i = 1$; and let D be the number of steps "down", i.e., the corresponding $W_i = -1$,

$$\left.\begin{aligned} U - D = h - X_0 \\ U + D = n \end{aligned}\right\} \Rightarrow U = \frac{n + h - X_0}{2}.$$

Then,

$$
\begin{aligned}
P\left(X_n = h\right) &= \binom{n}{U} p^U \left(1-p\right)^{n-U}, \\
&= \binom{n}{\frac{n+h-X_0}{2}} p^{\frac{n+h-X_0}{2}} \left(1-p\right)^{\frac{n-h+X_0}{2}}.
\end{aligned}
$$

Remark 1. *if $n >>>$(n is big enough), $X_n \sim N(.,.)$ by CLT.*

2.

$$
\begin{aligned}
E\left[X_n\right] &= E\left[X_0\right] + E\left[W_1\right] + \cdots + E\left[W_n\right], \\
&= X_0 + n\left(1 \times p + (-1)(1-p)\right), \\
&= X_0 + (2p-1)n.
\end{aligned}
$$

Leading to the following for different values of p:

$$
\begin{aligned}
&\text{if } p = \frac{1}{2}, \qquad E\left[X_n\right] = X_0, \\
&\text{if } p > \frac{1}{2}, \qquad E\left[X_n\right] \xrightarrow[n\to\infty]{} +\infty, \\
&\text{if } p < \frac{1}{2}, \qquad E\left[X_n\right] \xrightarrow[n\to\infty]{} -\infty.
\end{aligned}
$$

3.

$$
\begin{aligned}
V\left[X_n\right] &= V\left[X_0\right] + V\left[W_1\right] + \cdots + V\left[W_n\right], && (6) \\
&= 0 + 4np(1-p), && (7) \\
&= 4np(1-p). && (8)
\end{aligned}
$$

Where equation (6) is applicable because $W(n)$, $n = 1, 2, \ldots$, are i.i.d.

Remark 2. *By CLT, when $n \to \infty$,*

$$
\begin{aligned}
\lim_{n\to\infty} F_{X_n}(x_n) &= \int_{-\infty}^{x_n} \frac{\exp\left(-\frac{(x_n-(x_0+(2p-1)n))^2}{2(4np(1-p))}\right)}{\sqrt{2\pi}\sqrt{4np(1-p)}} dx_n, \\
&= \int_{-\infty}^{x_n} \frac{\exp\left(-\frac{(x_n-x_0)^2}{2n}\right)}{\sqrt{2\pi n}} dx_n, \qquad (for\ p = \frac{1}{2}).
\end{aligned}
$$

2 Brief review on the random walk process

Recall that the random walk process is a process that starts from a point $X_0 = h_0$. At each time instant n, $X_n = X_{n-1} \pm 1$ (c.f. fig. 3 for an example).

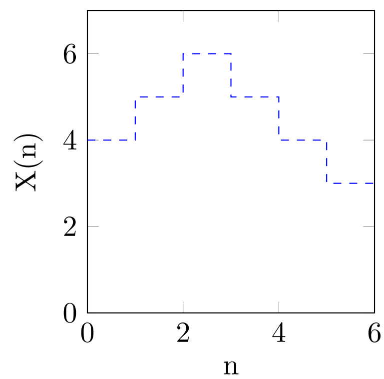


Figure 3: An example of the random walk process, $h_0 = 4$.

3 Independent Increments

Definition 3. *A Random Process is said to have independent increments if for all $n_1 < n_2 < \cdots < n_T$,*

$$X_{n_1}, X_{n_2} - X_{n_1}, \ldots, X_{n_T} - X_{n_{T-1}},$$

are jointly independent for all $T > 1$.

Example 3. *The random walk process is an independent increment process.*

For $T = 2$ and $n_1 < n_2$,

$$X_{n_1} = W_0 + W_1 + \cdots + W_{n_1},$$
$$X_{n_2} = W_0 + W_1 + \cdots + W_{n_1} + W_{n_1+1} + \cdots + W_{n_2}.$$

Hence,

$$X_{n_2} - X_{n_1} = W_{n_1+1} + W_{n_1+2} + \cdots + W_{n_2},$$

is independent of X_{n_1}, because the W_i are i.i.d.

Example 4. *Consider the random walk process X_n of last lecture where the W_i, $i = 1, 2, \ldots$ are i.i.d, $W_0 = h_0$, for $i \geq 1$*

$$W_i = \begin{cases} 1 & \textit{with probability } p = P(W_i = +1), \\ -1 & \textit{with probability } 1 - p = P(W_i = -1), \end{cases}$$

and $X_n = W_0 + W_1 + \cdots + W_n$.

Questions:

1. Find the probability $P(X_5 = a, X_7 = b)$.
2. Find the autocorrelation function $R_{XX}(k, l)$ of X_n.

Answers:

1. Recall from example 2 of last lecture that we can compute $P(X_n = h)$ using the binomial formula derived there. The first method to answer this question is by using Bayes' rule:

$$\begin{aligned} P(X_5 = a, X_7 = b) &= P(X_5 = a) P(X_7 = b | X_5 = a), \\ &= P(X_5 = a) P(X_7 - X_5 = b - a), \\ &= P(X_5 = a) P(X_2 - X_0 = b - a), \\ &= P(X_5 = a) P(X_2 = b - a + X_0), \end{aligned}$$

   And the second method is by using the independent increment property of the random walk process as follows:

$$\begin{aligned} P(X_5 = a, X_7 = b) &= P(X_5 = a, X_7 - X_5 = b - a), && (9) \\ &= P(X_5 = a) P(X_7 - X_5 = b - a). && (10) \end{aligned}$$

   Equation (10) follows from the independent increment property.

2. Now we can use this to find the autocorrelation function, assuming without loss of generality that $l > k$.

$$\begin{aligned} R_{XX}(k,l) &= E[x(k)^2], \\ &= E\left[(W_0 + W_1 + \cdots + W_k)^2\right], \\ &= E\left[W_0^2 + W_1^2 + \cdots + W_k^2 + 2(W_0 W_1 + \cdots + W_k W_{k-1})\right], \\ &= E\left[W_0^2\right] + kE\left[W_1^2\right] + 2h_0 (E[W_1] + \cdots + E[W_k]) + 2E[W_1 W_2 + \cdots + W_{k-1} W_k], \\ &= h_0^2 + k + 2h_0(2p-1) + 2\frac{k(k-1)}{2}(2p-1)^2. \end{aligned}$$

   For $p = \frac{1}{2}$,

$$R_{XX}(k,l) = \begin{cases} h_0^2 + k & l > k, \\ h_0^2 + l & l < k. \end{cases}$$

$$\Rightarrow R_{XX}(k,l) = h_0^2 + \min(k,l).$$

Practice 1. *Try to find at home $R_{XX}(k,l)$ in a different way, i.e., using*

$$R_{XX}(k,l) = [(W_1 + W_2 + \cdots + W_k)(W_1 + W_2 + \cdots + W_l)].$$

Definition 4. *A random process $X(t)$ is stationary if it has the same nth-order CDF as $X(t+T)$, that is, the two n-dimensional functions*

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + T, \ldots, t_n + T)$$

are identically equal for all T, for all positive integers n, and for all $t_1, \ldots, t_n$.

Example 5. *Consider the i.i.d. Gaussian r.p. $W_1, W_2, \ldots$ from last lecture.*

$$W_i \sim N(0, \sigma^2) \quad \forall\, i \geq 1.$$

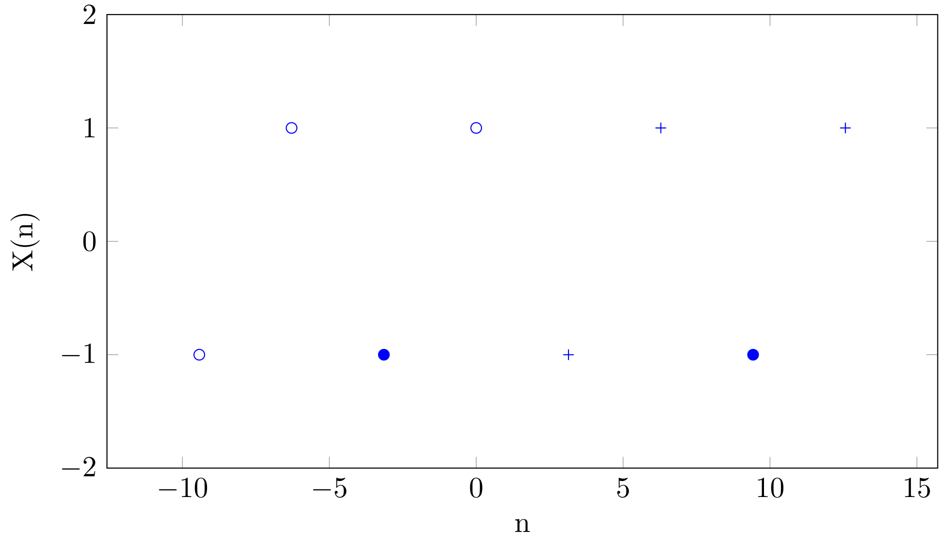


Figure 4: Plot of a stationary random process. The points having the same symbol, or any group of them, have the same distribution.

Question: Is this r.p. stationary?

Answer: This r.p. is stationary because:

1. All of the W_i, $i = 1, 2, \ldots$ have the same pdf, i.e., same mean and same variance.
2. Any two groups of them have the same jointly Gaussian distribution (since they are i.i.d).

Example 6. *Consider the averaging process, defined as*

$$X_i = \frac{W_i + W_{i-1}}{2}$$

Where W_i, $i = 0, 1, 2, \ldots$ are i.i.d Gaussian r.v.

Question: Is this r.p. stationary?

Answer:

1. The X_i,$i = 1, 2, \ldots$ have the same pdf since the W_i, $i = 0, 1, 2, \ldots$ are i.i.d.
2. Recall that we got the auto-correlation function in the previous lecture:

$$R_{XX}(k, l) = \begin{cases} \frac{\sigma^2}{2} & \text{if } k = l, \\ \frac{\sigma^2}{4} & \text{if } k = l \pm 1, \\ 0 & \text{if otherwise.} \end{cases}$$

It can be seen that there is correlation between X_i and X_j if and only if the distance between them is at most 1, i.e., $j = i-1, i, i+1$. For instance, consider $f_{X_1,X_4}(x_1, x_4)$, since $1 \neq 3, 4, 5$, thus X_1 and X_4 are independent and

$$\begin{aligned} f_{X_1,X_4}(x_1, x_4) &= f_{X_1}(x_1) f_{X_4}(x_4), \\ &= f_{X_{11},X_{14}}(x_{11}, x_{14}), \\ &= f_{X_{11}}(x_{11}) f_{X_{14}}(x_{14}). \end{aligned}$$

Moreover, even though X_1 and X_2 are correlated we can still say

$$f_{X_1,X_2}(x_1, x_2) = f_{X_{11},X_{12}}(x_{11}, x_{12}),$$

because they are both jointly Gaussian r.v. and have the same covariance matrices. Therefore, the averaging process is stationary.

This leads us to a more interesting case: the stationarity of the random walk process.

Example 7. *Consider the random walk process $X(n)$ defined as*

$$X_n = \begin{cases} h_0 & n = 0, \\ h_0 + W_1 + W_2 + \cdots + W_n & n \geq 1. \end{cases}$$

Where $h_0 = W_0 = X_0 =$ constant and W_i, $i = 1, 2, \ldots$ are Bernoulli r.v. that can take the values ± 1 with probability p and $1-p$.

Question: Is the random walk process stationary?

Answer: By simply looking at the mean of X_m for any m,

$$E[X_m] = (2p-1)m + X_0,$$

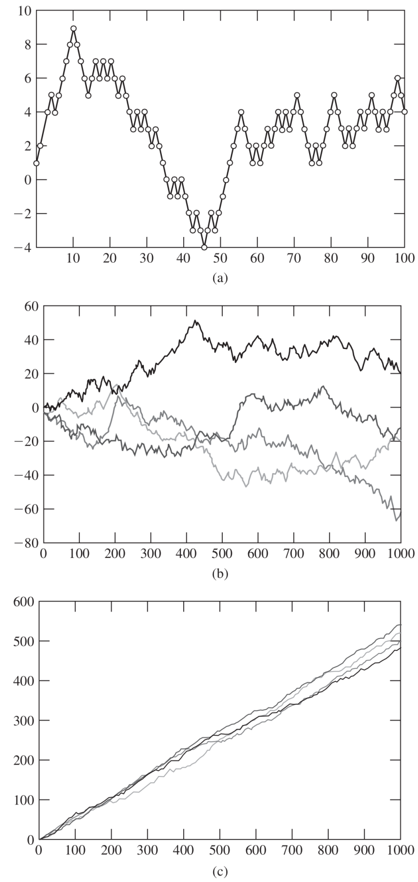
we see that for $p \neq 0.5$, $E[X_n]$ is a function of m. This means that the mean of each r.v. varies with m. Thus, the pdf of two different points can not be the same. For $p = 0.5$, although the mean is the same, the variance $V(X_n) = 4np(1-p)$ is increasing with time, hence two different points have different distributions. Therefore, the random walk process is not stationary. Interpretation can be found in Figure 5.

Definition 5. *$X(n)$ is called wide sense stationary (WSS) process, iff:*

1. *$E[X(n)] = E[X(0)]$, $\forall$ n (average does not change with time).*
2. *$R_{XX}(k, l) = R_{XX}(k+n, l+n) = R_X(k-l)$ (the autocorrelation function depends only on the time difference).*

Example 8. *Is a random walk process WSS?*

In general, $E(X_n) = (2p-1)n + X_0$ changes with n. So, it does not satisfy the mean condition. Also, since $R_{XX} = h_0^2 + \min(k, l)$ changes with k and l, thus, it fails the auto-correlation condition either. Therefore, the random walk process is not WSS.



(a) Random walk process with $p = 1/2$. (b) Four sample functions of symmetric random walk process with $p = 1/2$. (c) Four sample functions of asymmetric random walk with $p = 3/4$.

Figure 5

Lemma 1. *If a random process $X(n)$ is stationary, then it also is wide sense stationary.*

$$X(n) \text{ stationary} \Rightarrow X(n) \text{ is WSS.}$$

Lemma 2. *If $X(n)$ is Gaussian r.p., then if $X(n)$ is wide sense stationary, it is also stationary.*

$$X(n) \text{ is Gaussian and WSS} \Leftrightarrow X(n) \text{ is stationary.}$$

Example 9. *Let Θ be a R.V. uniformly distributed on $[-\pi, \pi]$ and $X(n) = \cos(n\omega + \Theta)$, where ω is a constant.*

Questions:

1. Is $X(n)$ WSS?
2. Is $X(n)$ stationary?

Answers:

1. To check the Mean Condition:

$$\begin{aligned}
E\left[X(n)\right] &= \int_{-\infty}^{+\infty} f_{\Theta}(\theta)\cos\left(n\omega+\theta\right)d\theta, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(n\omega+\theta\right)d\theta, \\
&= 0.
\end{aligned}$$

   It is equal to 0 because the cosine function is symmetric between $-\pi$ and π. $E\left[X(n)\right]$ is constant for any n, then it satisfies the mean condition.

2. To check the Auto-Correlation Condition:

$$\begin{aligned}
R_{XX}(k,l) &= E\left[X\left(k\right)X\left(l\right)\right], && (11)\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\cos\left(k\omega+\theta\right)\cos\left(l\omega+\theta\right)d\theta, && (12)\\
&= \frac{1}{2}\cos\left(\omega\left(k-l\right)\right), && (13)\\
&= R_{XX}(k-l). && (14)\\
& && (15)
\end{aligned}$$

   Where equation (13) can be found after some trigonometric calculations beginning by

$$\cos\alpha cos\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)+\cos\left(\alpha+\beta\right)\right].$$

   Thus, $X(n)$ is WSS.

3. $X(n)$ is stationary. We will not prove it rigourously, however we will give the main idea. Consider $\omega=\pi$,

$$\begin{aligned}
X(n) &= cos(n\pi+\Theta),\\
X(1) &= cos(\pi+\Theta),\\
X(2) &= cos(2\pi+\Theta) = cos(\Theta),\\
X(3) &= cos(3\pi+\Theta) = cos(\pi+\Theta),\\
X(4) &= cos(4\pi+\Theta) = cos(\Theta).
\end{aligned}$$

   It is clear that any X_n is uniformly distributed on $[-1,1]$ because the cosine is symmetric on any interval of length 2π. in particular for $\Theta \sim U[-\pi,\pi]$ and $\Theta' = \Theta+\pi \sim U[0,2\pi]$, $\cos(\Theta) \sim U[-1,1]$ and $\cos(\Theta') \sim U[-1,1]$. Therefore, the X_n have same distribution and $X(n)$ is stationary.

   Generally, since $\Theta \sim U[-\pi,\pi]$ and ω is a constant. The r.v. $n\omega+\Theta$ is uniformly distributed on $[n\omega-\pi, n\omega+\pi]$ which is an interval of length $2\pi \Rightarrow \cos\left(n\omega+\Theta\right) \sim U[-1,1]$ for all n. Therefore, all the X_n have the same distribution and $X(n)$ is stationary.

4 Continuous Time Random Process

Definition 6. *A continuous time random process $X(t)$, is a random process defined for any $t \in \mathbb{R}$.*

We can redefine, in continuous time, everything defined in discrete time.

5 Poisson process

The Poisson process is a special case of a counting process. So first, we will start by defining a counting process.

5.1 Counting process

Definition 7 (Counting process)**.** *A counting process is a continuous random process $\{N(t), t \geq 0\}$ with values that are non-negative, integer, and non-decreasing:*

1. $N(t) \geq 0$.
2. $N(t)$ *is an integer.*
3. *If $s \leq t$ then $N(s) \leq N(t)$.*

Example 10 (Bernoulli process)**.** *Let $X_i, i = 1, \ldots, n$ be iid Bernoulli random variables with parameter p. Define the Bernoulli process $S_n = X_1 + \ldots + X_n$. The Bernoulli process is a discrete counting process because it satisfies the three conditions in Definition 7. Namely,*

- *Since $X_i \in \{0, 1\}, i = 1, \ldots, n$, then $X_i \geq 0$ and X_i is an integer. Therefore, $S_n = X_1 + \ldots + X_n$ is a non-negative integer.*
- *For $m \leq n$, we have $S_n - S_m = X_{m+1} + \ldots + X_n \geq 0$. Hence, if $m \leq n$ then $S_m \leq S_n$.*

Now, we define the Poisson process.

Definition 8 (Poisson process)**.** *The Poisson process is a counting process $\{N(t), t \geq 0\}$ that satisfies the following three properties:*

1. $N(0) = 0$.
2. *$N(t)$ has independent incerements, i.e., for $t_0 < t_1 < \ldots < t_n$, the random variables $(N(t_1) - N(t_0)), (N(t_2) - N(t_1)), \ldots, (N(t_n) - N(t_{n-1}))$ are independent.*
3. *The total count in any interval of length t is a Poisson random variable with parameter (or mean) λt, i.e.,*

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \; k = 0, 1, 2, \ldots.$$

It follows from the Definition 8 that

$$\mu_N(t) = E[N(t)] = \lambda t,$$

$$V(N(t)) = \lambda t.$$

Moreover, based on the independent increments property we have

$$\begin{aligned} P(N(t_1) = i, N(t_2) = j) &= P(N(t_1) = i)\, P(N(t_2) = j | N(t_1) = i), \\ &= P(N(t_1) = i)\, P(N(t_2) - N(t_1) = j - i), \\ &= P(N(t_i) = i)\, P(N(t_2 - t_1) = j - i), \\ &= \frac{(t_1\lambda)^i e^{-t_1\lambda}}{i!} \frac{((t_2 - t_1)\lambda)^{j-i}\, e^{-(t_2-t_1)\lambda}}{(j-i)!}. \end{aligned}$$

5.2 Interarrival times

We think of the Poisson process $N(t)$ as a process that counts arrivals in a time interval of length t. For example, this process can be used to count the number customers that arrive to a certain store within t seconds. These customers arrive independently at a rate of λ customers/second. Let $S_i, i = 1, 2, \ldots$, be the random variable that represents the time of the i^{th} arrival, i.e.,

$$S_i \triangleq \inf\{t \geq 0 : N(t) = i\}, \quad i = 1, 2, \ldots.$$

The interarrival times are given by

$$X_i = S_i - S_{i-1}, \quad i = 1, 2, \ldots.$$

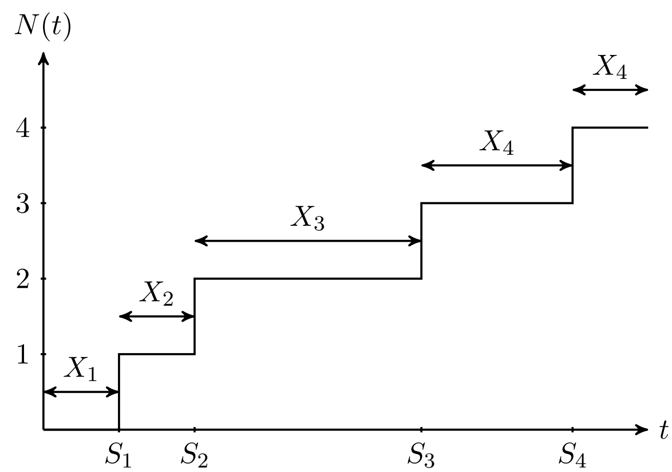


Figure 6: A realization of the Poisson random process. The X_i's represent the interarrival times.

Theorem 1. *The first arrival time $X_1 \sim \exp(\lambda)$.*

Proof.

$$P(X_1 > t) = P(N(t) = 0) = \frac{(t\lambda)^0)e^{-t\lambda}}{0!} = e^{-t\lambda}.$$

Therefore,

$$F_{X_1}(t) = P\left(X_1 \leq t\right) = 1 - e^{-\lambda t}.$$

This is the CDF of an $\exp(\lambda)$ random variable. $\square$

Theorem 2. *All the interarrival times, X_i, $i = 1, 2, \ldots,$ are iid and have a distribution* $\exp(\lambda)$.

Proof.

$$P\left(X_{n+1} > t | X_1 = t_1, \ldots, X_n = t_n\right) = P\left(X_{n+1} > t | S_1 = s_1, \ldots, S_n = s_n\right),$$

where $s_i = t_1 + \ldots + t_i$, for $i = 1, \ldots, n$. Therefore,

$$\begin{aligned}
P\left(X_{n+1} > t | X_1 = t_1, \ldots, X_n = t_n\right) &= P\left(X_{n+1} > t | S_1 = s_1, \ldots, S_n = s_n\right) \\
&= P\left(S_{n+1} > t + s_n | S_n = s_n\right) \\
&= P\left(N(t + s_n) - N(s_n) = 0 | S_n = s_n\right) \\
&= P\left(N(t) = 0 | S_n = s_n\right) \ \text{(independent increments)} \\
&= P\left(N(t) = 0\right) \\
&= e^{-\lambda t} \\
&= Pr(X_1 > t).
\end{aligned}$$

Therefore, $X_i, i = 1, 2, \ldots$ are iid and have a distribution $\exp(\lambda)$. $\square$

5.3 Autocovariance and stationarity of Poisson process

Assume, without loss of generality, that $t_1 < t_2$.

$$\begin{aligned}
K_{NN}\left(t_1, t_2\right) &= E\left[\left(N(t_1) - \lambda t_1\right)\left(N(t_2) - \lambda t_2\right)\right], \\
&= E\left[\left(N(t_1) - \lambda t_1\right)\left(N(t_2) - N(t_1) - \lambda t_2 + \lambda t_1 + N(t_1) - \lambda t_1\right)\right], \\
&= \underbrace{E\left[\left(N(t_1) - \lambda t_1\right)\left(N(t_2) - N(t_1)\right) - \left(\lambda t_2 - \lambda t_1\right)\right]}_{0} + E\left[\left(N(t_1) - \lambda t_1\right)^2\right], \\
&= \lambda t_1.
\end{aligned}$$

If $t_2 \leq t_1$ then $K_{NN}\left(t_1, t_2\right) = \lambda t_2$. In general, $K_{NN}\left(t_1, t_2\right) = \lambda \min\left(t_1, t_2\right)$.

Question: Is $N(t)$ WSS?

Answer:

1. $E\left[N(t)\right] = \lambda t$, depends on t.
2. $K_{NN}\left(t_1, t_2\right) = \lambda \min\left(t_1, t_2\right)$, depends on t.

Hence $N(t)$ is not WSS.

6 Continuous Gaussian Random Process

Definition 9. *$X(t)$ is a Gaussian random process if $X(t_1), X(t_2), \ldots, X(t_k)$ are jointly Gaussian for any k, i.e.,*

$$f_{X(t_1),X(t_2),\ldots,X(t_k)} = \frac{1}{(2\pi)^{\frac{1}{2}} |K_{XX}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\underline{X} - \underline{\mu}_X\right)^T K_{XX}^{-1}\left(\underline{X} - \underline{\mu}_x\right)\right).$$

Example 11. *Let $X(t)$ be a Gaussian random process with $\mu_X(t) = 3t$ and $K_{XX} = 9e^{-2|t_1 - t_2|}$.*

Question: Find the pdf of $X(3)$ and $Y = X(1) + X(2)$.

Answer: We know that $X(3)$ is a Gaussian r.v. (by definition of $X(t)$) and Y is also a Gaussian r.v. being a linear combination of two Gaussian r.v. Therefore, it is enough to find the mean and the variance of those variables in order to find their PDFs.

1. *$X(3)$:* $\mu_X(3) = 9$ and $V[X(3)] = 9e^{-|3-3|} = 9$. Thus,

$$f_{X(3)}(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-9)^2}{9}}.$$

2. *Y:*

$$\begin{aligned} E[y] &= E[X(1)] + E[x(2)] = 3 + 6 = 9, \\ V[Y] &= V(X(1)) + V[X(2)] + 2cov(X(1), X(2)), \\ cov(X(1), X(2)) = 9e^{-2|2-1|} &= 9e^{-2}, \\ V[Y] &= 9 + 9 + 9e^{-2} = 18 + 18e^{-2}. \end{aligned}$$

Another way to find the variance is the following:

$$\begin{aligned} V[Y] &= E\left[(X(1) + X(2))^2\right] - (E[X(1)] + E[X(2)])^2, \\ &= E\left[X(1)^2\right] + E\left[X(2)^2\right] + 2E[X(1)X(2)] - E^2[X(1)] - E^2[X(2)] - 2E[X(1)X(2)], \\ &= V(X(1)) + V(X(2)) + 2cov(X(1), X(2)), \\ &= 9 + 9 + 2 \times 9e^{-2}, \\ &= 18 + 18e^{-2}. \end{aligned}$$