ECE541: Stochastic Signals and Systems Fall 2018

Chapter 3 : Functions of Random Variables

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1 Functions of Random Variables of the Type $Y = g(X)$

Example 1. *Let X be a uniform RV on $(0,1)$, that is, $X : U(0,1)$, and let $Y = 2X + 3$. What is the pdf of Y?*

$$\begin{aligned} F_Y(y) &= Pr(Y \leq y) \\ &= Pr(2X + 3 \leq y) \\ &= Pr\left(X \leq \frac{y-3}{2}\right) \\ &= F_X\left(\frac{y-3}{2}\right). \\ f_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{1}{2} f_X\left(\frac{y-3}{2}\right). \end{aligned}$$

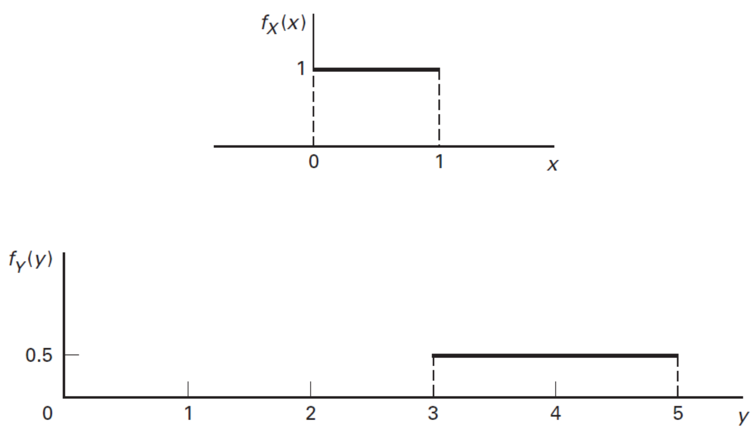


Figure 1: PDF of X and Y.

Generalization: Let $Y = aX + b$, where a ($a \neq 0$) and b are certain constants and X is continuous RV with pdf $f_X(x)$. Then the pdf of Y is given by:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Example 2. *Let X be a RV with continuous CDF $F_X(x)$ and let $Y = X^2$. What is the pdf of Y?*

$$\begin{aligned} F_Y(y) &= Pr(Y \leq y) \\ &= Pr(X^2 \leq y). \end{aligned}$$

For $y \geq 0$,

$$
\begin{aligned}
F_Y(y) &= Pr(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= F_X(+\sqrt{y}) - F_X(-\sqrt{y}) + Pr(X = -\sqrt{y}). \\
f_Y(y) &= \frac{dF_Y(y)}{dy} \\
&= \frac{dF_X(+\sqrt{y})}{d(\sqrt{y})}\frac{d(\sqrt{y})}{dy} - \frac{dF_X(-\sqrt{y})}{d(-\sqrt{y})}\frac{d(-\sqrt{y})}{dy} \\
&= f_X(\sqrt{y}) \times \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \times \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})).
\end{aligned}
$$

Suppose that $X \sim N(0,1)$, then the pdf of Y in this case is given by:

$$
f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} & \text{if } y \geq 0, \\ 0 & \text{if } y < 0. \end{cases}
$$

Theorem 1. *Given a continuous RV X with pdf $f_X(x)$, and a differentiable function $g(X)$. The pdf of $Y = g(X)$ is given by,*

$$
f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|},
$$

where the x_i's, $i = 1, \ldots, n$, are the roots of $y = g(x)$ and $g'(x_i) \neq 0$.

Example 3. *Let X be a random variable uniformly distributed over $(-\pi, +\pi)$ and let $Y = \sin X$. What is the pdf of Y?*
In this example, $g(x) = \sin x$ and $g'(x) = \cos x$. The equation $g(x) = y$ has two roots for $|y| < 1$, which are given by $x_1 = \sin^{-1} y$ and $x_2 = \pi - \sin^{-1} y$. By applying Theorem 1,

$$
\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\
&= \frac{1}{2\pi}\frac{1}{|\cos(\sin^{-1} y)|} + \frac{1}{2\pi}\frac{1}{|\cos(\pi - \sin^{-1} y)|} \\
&= \frac{1}{\pi}\frac{1}{|\cos(\sin^{-1} y)|}.
\end{aligned}
$$

To evaluate $\cos(\sin^{-1} y)$ we make use of figure 2.

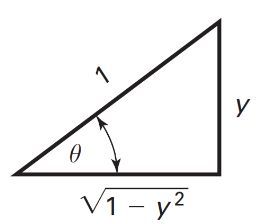


Figure 2: Evaluating $\cos(\sin^{-1} y)$.

As shown in figure 2, $\theta = \sin^{-1} y$ *and* $\cos\theta = \sqrt{1-y^2} = \cos(\sin^{-1} y)$*. Hence,*

$$f_Y(y) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} & \text{if } |y| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4. *A student at a train station awaits the arrival of either a red or a green train. At this station, red and green trains arrive independently with a rate* $\lambda_r = 0.1$ *train/min for red trains and a rate of* $\lambda_g = 0.5$ *trains/min for green trains. Let* T_R *be the time the student waits until a red train arrives, and* T_G *be the time the students waits until a green train arrives. Given* $T_G \sim exp(\lambda_g)$ *and* $T_R \sim exp(\lambda_r)$*.*

1. *What is the probability that the green train arrives first?*

$$\begin{aligned} Pr(T_G < T_R) &= \int_0^{+\infty} Pr(T_R > t | T_G = t) f_{T_G}(t) dt \\ &= \int_0^{+\infty} (1 - Pr(T_R \leq t | T_G = t)) f_{T_G}(t) dt \\ &= \int_0^{+\infty} (1 - F_{T_R}(t)) f_{T_G}(t) dt \\ &= \int_0^{+\infty} e^{-\lambda_r t}\ \lambda_g\ e^{-\lambda_g t} dt \\ &= \frac{\lambda_g}{\lambda_g + \lambda_r} \\ &= \frac{5}{6}. \end{aligned}$$

2. *Let* T *be the time the student waits until a red or a green train arrives. What is the pdf of* T*? Intuitively,* T *can be expressed as*

$$T = min\{T_R, T_G\}.$$

Therefore for $t \geq 0$*,*

$$\begin{aligned} Pr(T \leq t) &= Pr(min\{T_R, T_G\} \leq t) \\ &= 1 - Pr(min\{T_R, T_G\} > t) \\ &= 1 - Pr(T_R > t, T_G > t) \\ &= 1 - Pr(T_R > t) Pr(T_G > t) \\ &= 1 - (1 - Pr(T_R \leq t))(1 - Pr(T_G \leq t)) \\ &= 1 - (1 - F_{T_R}(t))(1 - F_{T_G}(t)) \\ &= 1 - e^{-(\lambda_r + \lambda_g)t}. \end{aligned}$$

Therefore, $T \sim exp(\lambda_r + \lambda_g) = exp(0.6)$*.*

2 Functions of Random Variables of the Type $Z = g(X, Y)$

Theorem 2. *Given two* independent *random variables X and Y with pdfs $f_X(x)$ and $f_Y(y)$ respectively, the pdf of $Z = X + Y$ is given by,*

$$f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{+\infty} f_X(y) f_Y(z - y) dy.$$

Example 5. *Let X and Y be independent random variables such that $X \sim exp(1)$ and $Y \sim Uniform(-1, 1)$, and let $Z = X + Y$. What is the pdf of Z?*

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(y) f_Y(z - y) dy.$$

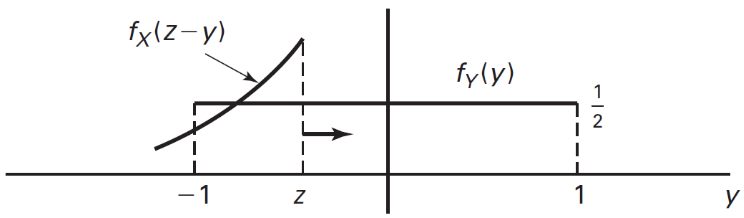


Figure 3: Relative positions of $f_X(z - y)$ and $f_Y(y)$.

1. *If $z \leq -1$,*

$$f_Z(z) = 0.$$

2. *If $-1 \leq z \leq 1$,*

$$f_Z(z) = \frac{1}{2} \int_{-1}^{z} e^{-(z-y)} dy = \frac{1}{2} \left(1 - e^{-1-z}\right).$$

3. *If $z \geq 1$,*

$$f_Z(z) = \frac{1}{2} \int_{-1}^{1} e^{-(z-y)} dy = \frac{1}{2} \left(e^{1-z} - e^{-1-z}\right).$$

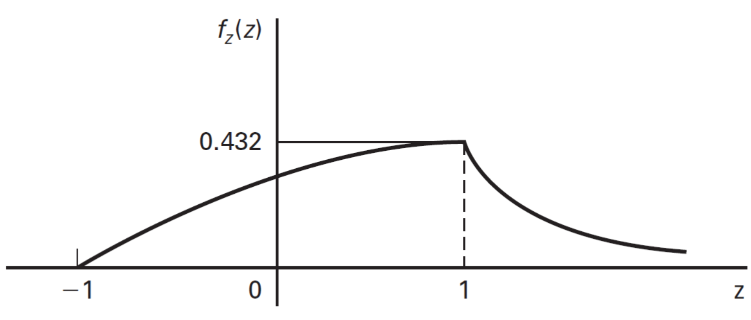


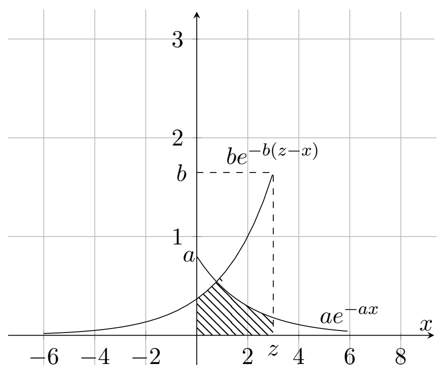
Figure 4: The pdf $f_Z(z)$.

Example 6. *Let X and Y be independent random variables such that $X \sim exp(a)$ and $Y \sim exp(b)$, and let $Z = X + Y$. What is the pdf of Z?*

$$f_X(x) = \begin{cases} ae^{-ax} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} be^{-by} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(y) f_Y(z-y) dy.$$



1. If $z \leq 0$,

$$f_Z(z) = 0.$$

2. If $z \geq 0$,

$$\begin{aligned} f_Z(z) &= \int_0^z abe^{-ay}e^{-b(z-y)}dy \\ &= abe^{-bz}\int_0^z e^{(b-a)y}dy. \end{aligned}$$

Therefore, for $z \geq 0$,

$$f_Z(z) = \begin{cases} \frac{ab}{a-b}\left(e^{-bz} - e^{-az}\right) & \text{if } a \neq b, \\ abze^{-bz} & \text{if } a = b. \end{cases}$$

Example 7. *Let X and Y be two iid (independent and identically distributed) random variables such that $X, Y \sim N(0,1)$.*

1. *What is the pdf of $Z = X^2 + Y^2$?*

Method 1:

$$Z = T + W$$
$$T = X^2$$
$$W = Y^2$$

Since T *and* W *are independent,* $f_Z = f_T * f_W$.

Method 2:

$$\begin{aligned} F_Z(z) &= Pr(Z \le z) \\ &= Pr(X^2 + Y^2 \le z) \\ &= \iint_{x^2+y^2 \le z} f_{X,Y}(x,y)dxdy \\ &= \iint_{x^2+y^2 \le z} f_X(x) f_Y(y)dxdy \\ &= \frac{1}{2\pi} \iint_{x^2+y^2 \le z} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} dxdy \\ &= \frac{1}{2\pi} \iint_{x^2+y^2 \le z} e^{-\frac{x^2+y^2}{2}} dxdy. \end{aligned}$$

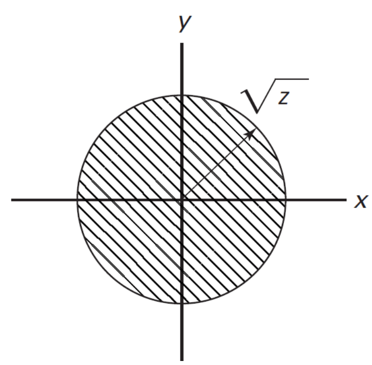


Figure 5: The region of the event $\{X^2 + Y^2 \le z\}$ for $z \ge 0$.

We evaluate this integral by transforming to polar coordinates,

$$x = r\cos\theta.$$
$$y = r\sin\theta.$$
$$x^2 + y^2 = r^2.$$

Therefore,

$$F_Z(z) = \int_0^{\sqrt{z}} \int_0^{2\pi} \frac{e^{-\frac{r^2}{2}}}{|J|} d\theta dr.$$

Where $|J|$ is determinant of the Jacobian of the transformation and is given by,

$$J = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix}$$

$|J|$ is also equal to the inverse of the Jacobian of the inverse transformation

$$|J|^{-1} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Therefore, for $z \geq 0$,

$$\begin{aligned} F_Z(z) &= \frac{1}{2\pi}\int_0^{\sqrt{z}}\int_0^{2\pi} re^{-\frac{r^2}{2}}\,d\theta dr \\ &= \frac{1}{2\pi}\int_0^{\sqrt{z}} re^{-\frac{r^2}{2}}\,dr \int_0^{2\pi} d\theta \\ &= 1 - e^{-\frac{z}{2}}. \\ f_Z(z) &= \frac{1}{2}e^{-\frac{z}{2}}. \end{aligned}$$

Therefore, $Z \sim exp(0.5)$.

2. *What is the pdf of $Z' = \sqrt{X^2 + Y^2}$?*

$$\begin{aligned} F_{Z'}(z') &= Pr(Z' \leq z') \\ &= Pr(Z'^2 \leq z'^2) \\ &= Pr(Z \leq z'^2) \\ &= F_Z(z'^2) \\ &= 1 - e^{-\frac{z'^2}{2}}. \\ f_{Z'}(z') &= z'e^{-\frac{z'^2}{2}}. \text{ (Rayleigh distribution)} \end{aligned}$$

Example 8. *Let X and Y be two iid random variables such that $X, Y \sim N(0,1)$, and let $Z = Y/X$. What is the pdf of Z?*
By conditioning on X (fixing) and applying the general linear transformation we get,

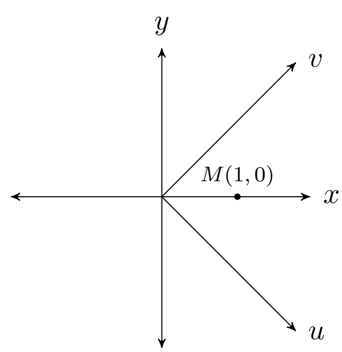
$$f_{Z|X=x}(z|X = x) = \frac{|x|}{\sqrt{2\pi}}e^{-\frac{x^2z^2}{2}}.$$

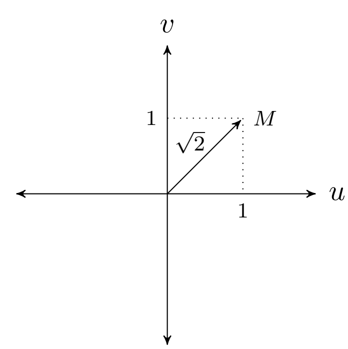
Therefore, by applying the total law of probability,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_{Z|X=x}(z|X=x) f_X(x) dx \\ &= \int_{-\infty}^{+\infty} \frac{|x|}{\sqrt{2\pi}} e^{-\frac{x^2z^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |x| e^{-\frac{x^2(1+z^2)}{2}} dx \\ &= \frac{1}{\pi} \frac{1}{1+z^2}. \end{aligned}$$

3 Functions of Random Variables of the Type $U = g(X,Y)$ and $V = h(X,Y)$

Example 9. *Let X and Y be two iid random variables such that $X, Y \sim N(0,1)$. Let $U = X+Y$ and $V = X - Y$. What is joint pdf of U and V?*
Consider the point M shown in the figures below.





This figures illustrate a case of a one-to-one mapping, because the linear system of equations

$$\begin{cases} X+Y &= 1 \\ X-Y &= 1 \end{cases}$$

is invertible, i.e. $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0.$
In fact,

$$f_{U,V}(1,1) = \frac{f_{X,Y}(1,0)}{|J|}.$$

Where,

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

$$\Rightarrow f_{U,V}(1,1) = \frac{f_{X,Y}(1,0)}{2}.$$

Generalizing,

$$f_{U,V}(u,v) = \frac{f_{X,Y}(x,y)}{|J|},$$

such that,

$$x = \frac{u+v}{2},$$
$$y = \frac{u-v}{2}.$$

Therefore,

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2} f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \\ &= \frac{1}{4\pi} \exp\left[-\frac{1}{8}\left[(u+v)^2 - (u-v)^2\right]\right]. \end{aligned}$$

Theorem 3. *Given two continuous RVs X and Y with pdfs $f_X(x)$ and $f_Y(y)$ respectively, and two differentiable functions $g_1(x)$ and $g_2(x)$. The joint pdf of $U = g_1(X,Y)$ and $V = g_2(X,Y)$ is given by,*

$$f_{U,V}(u,v) = \sum_{i=1}^{n} \frac{f_{X,Y}(x_i, y_i)}{|J(x_i, y_i)|},$$

where the pairs (x_i, y_i), $i = 1, \ldots, n$, are the solutions of the system of equations given by,

$$\begin{cases} g_1(x,y) = u \\ g_2(x,y) = v \end{cases} .$$