What Hockey Teams and Foraging Animals Can Teach Us About Feedback Communication Part II: Timid/Bold Coding

Aaron Wagner
Cornell University

In collaboration with
Nirmal Shende and Yücel Altuğ

On Feedback

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  - [Other contexts: networks, control over noisy channels, streaming codes, complexity-constrained coding....]

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Is this it?

Discrete Memoryless Channels without Feedback

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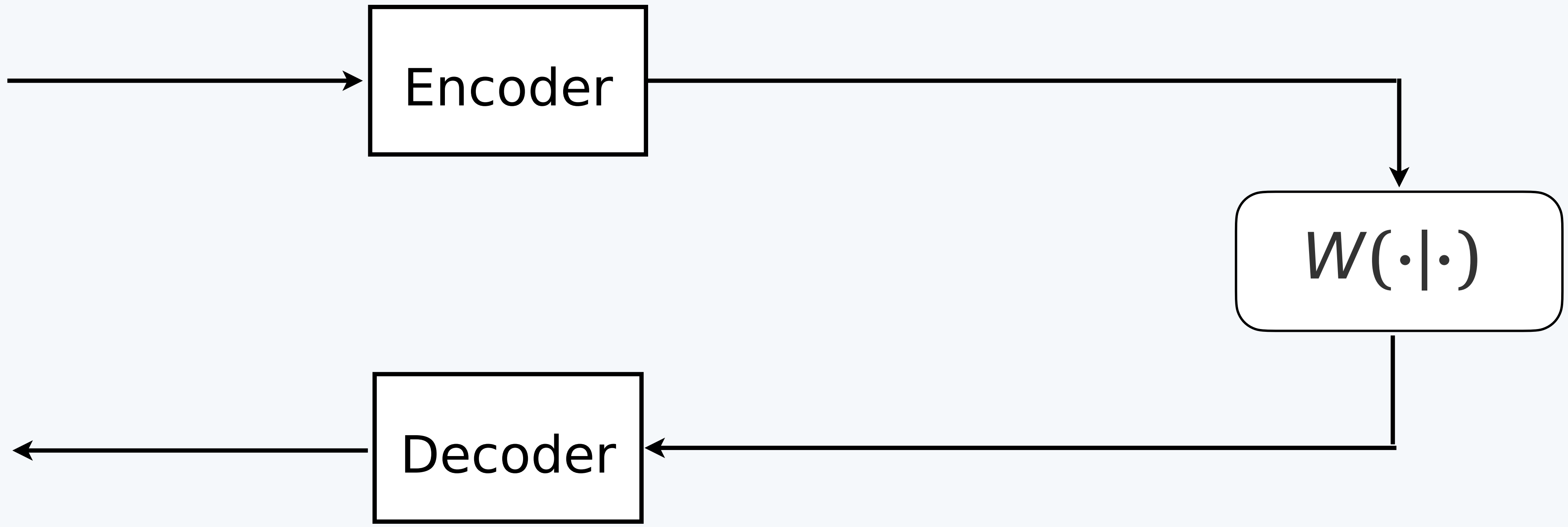

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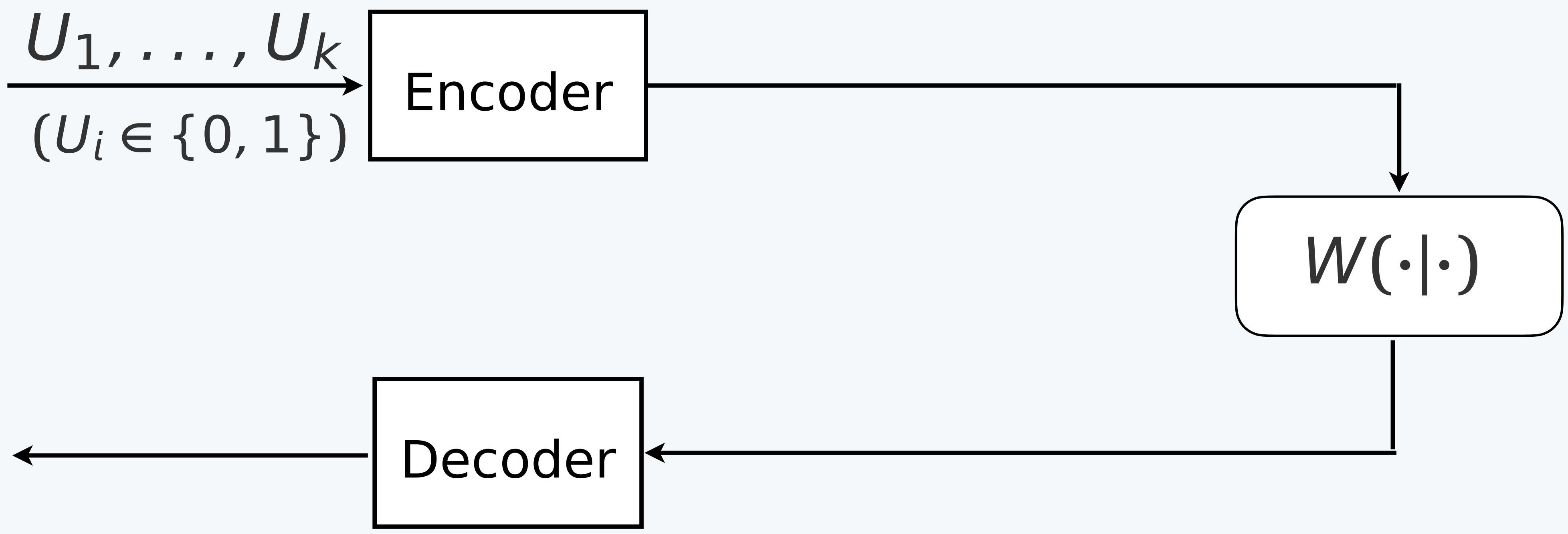

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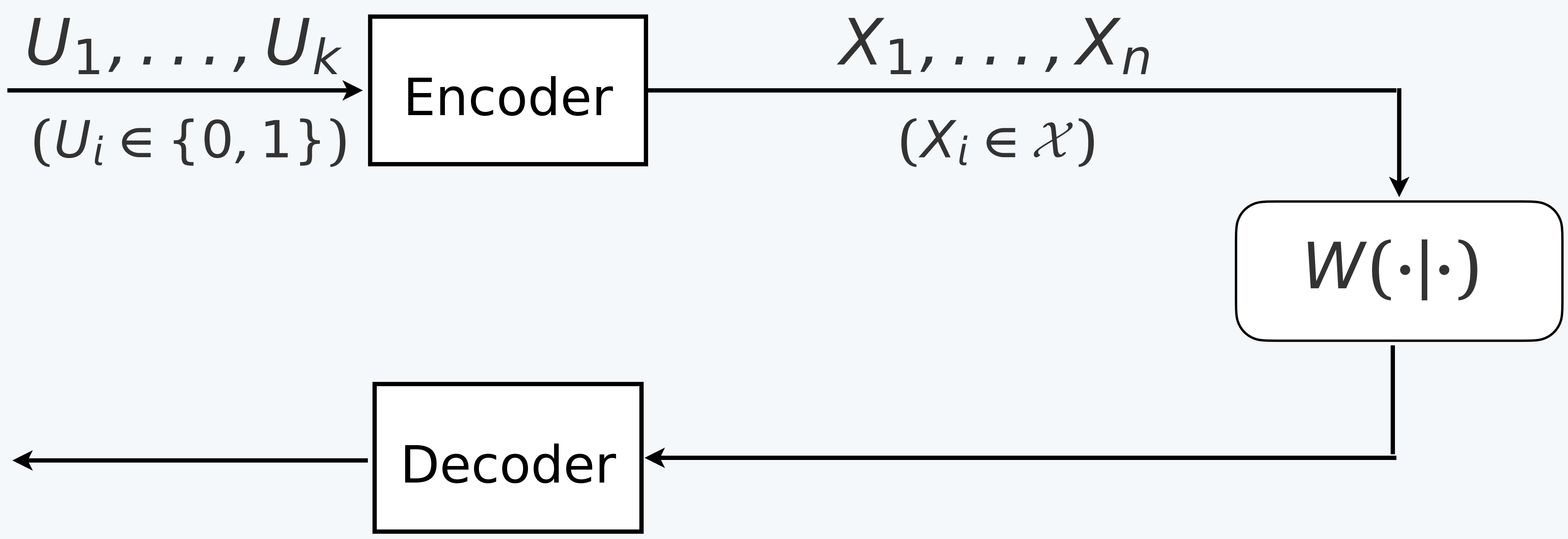

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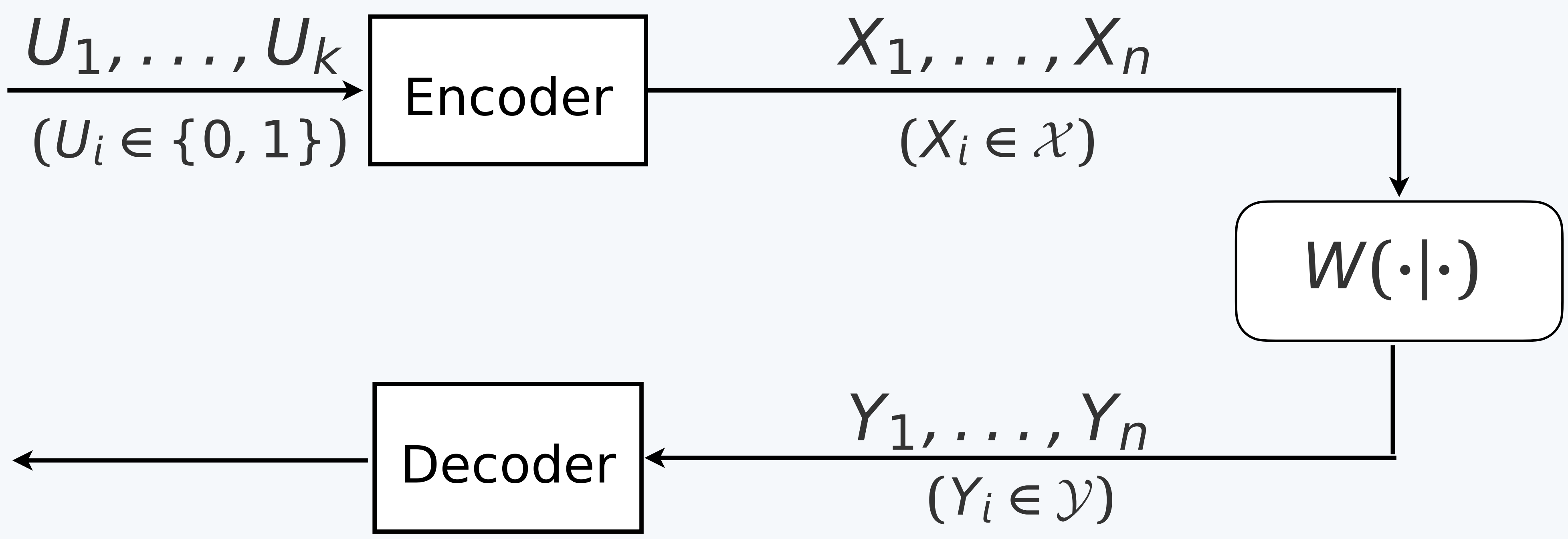

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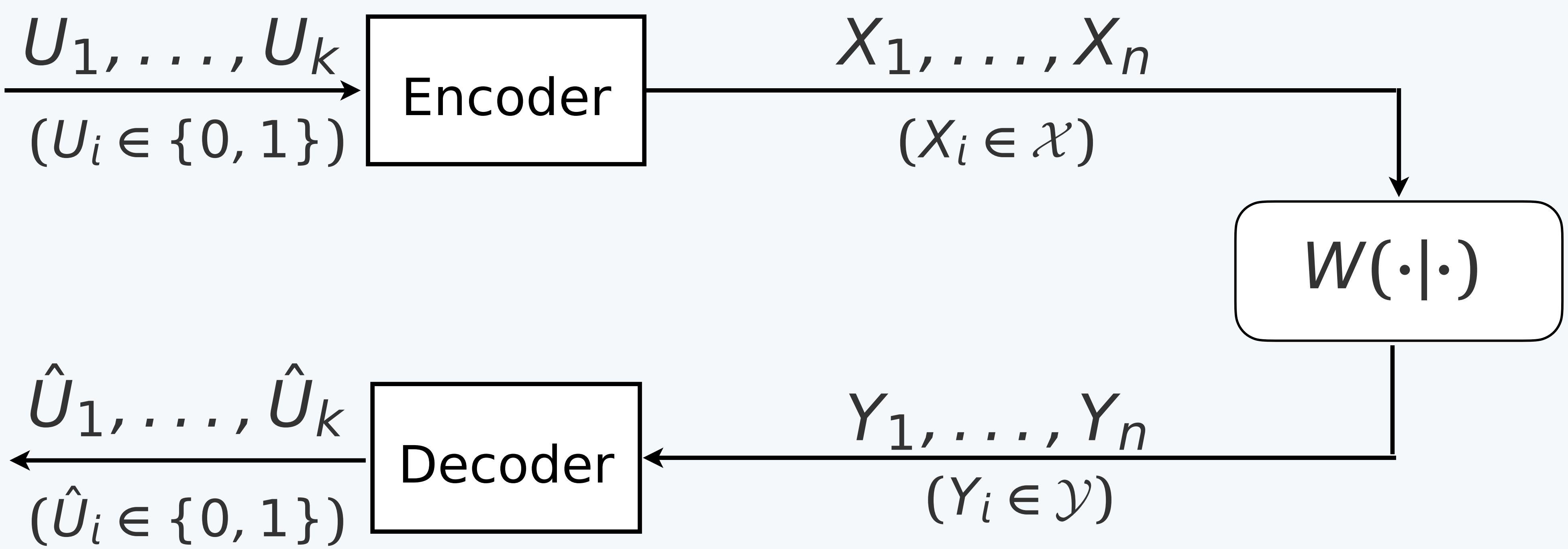

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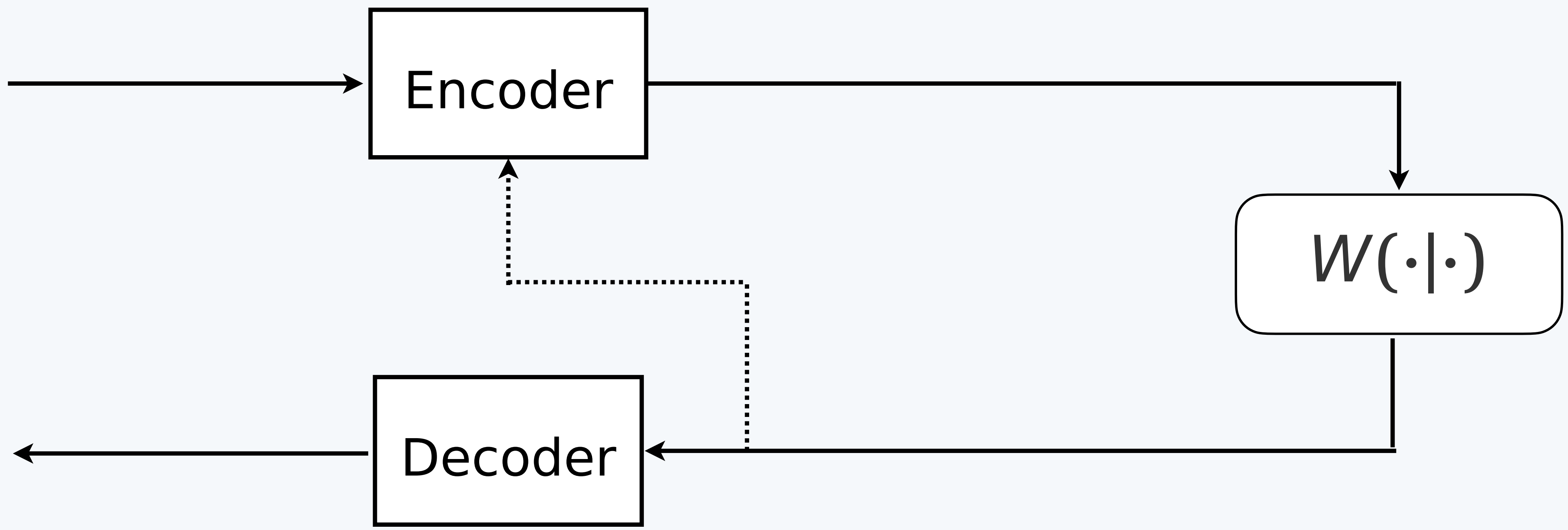

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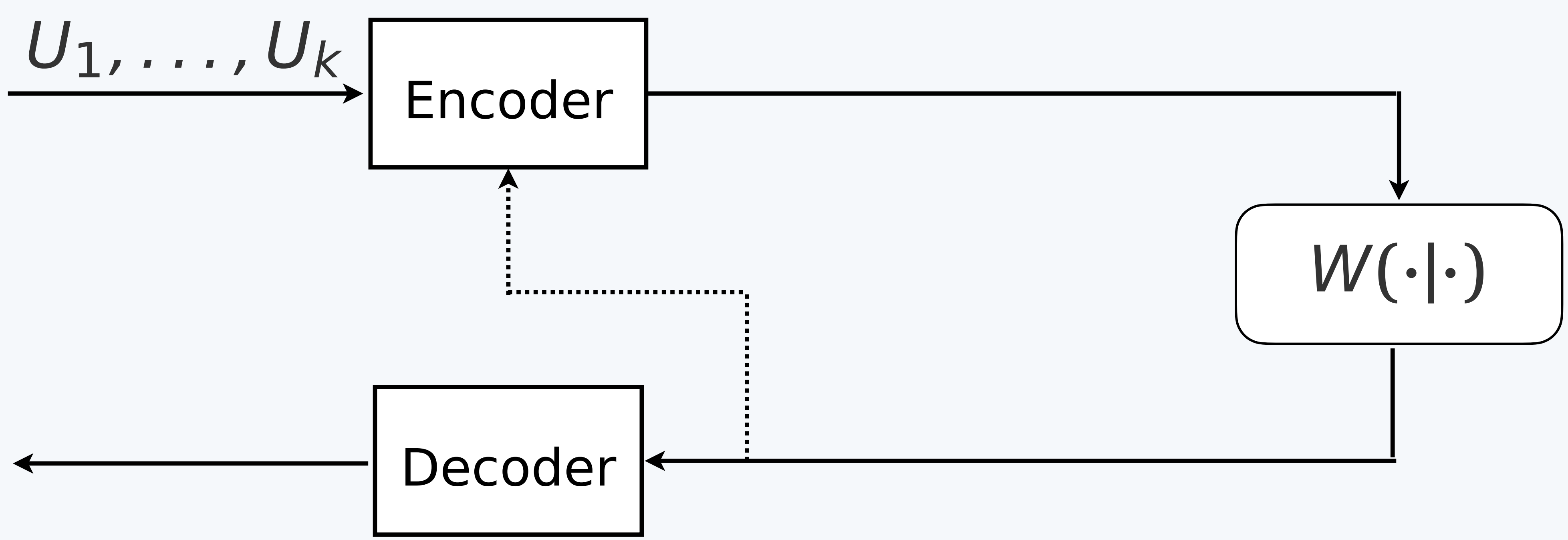

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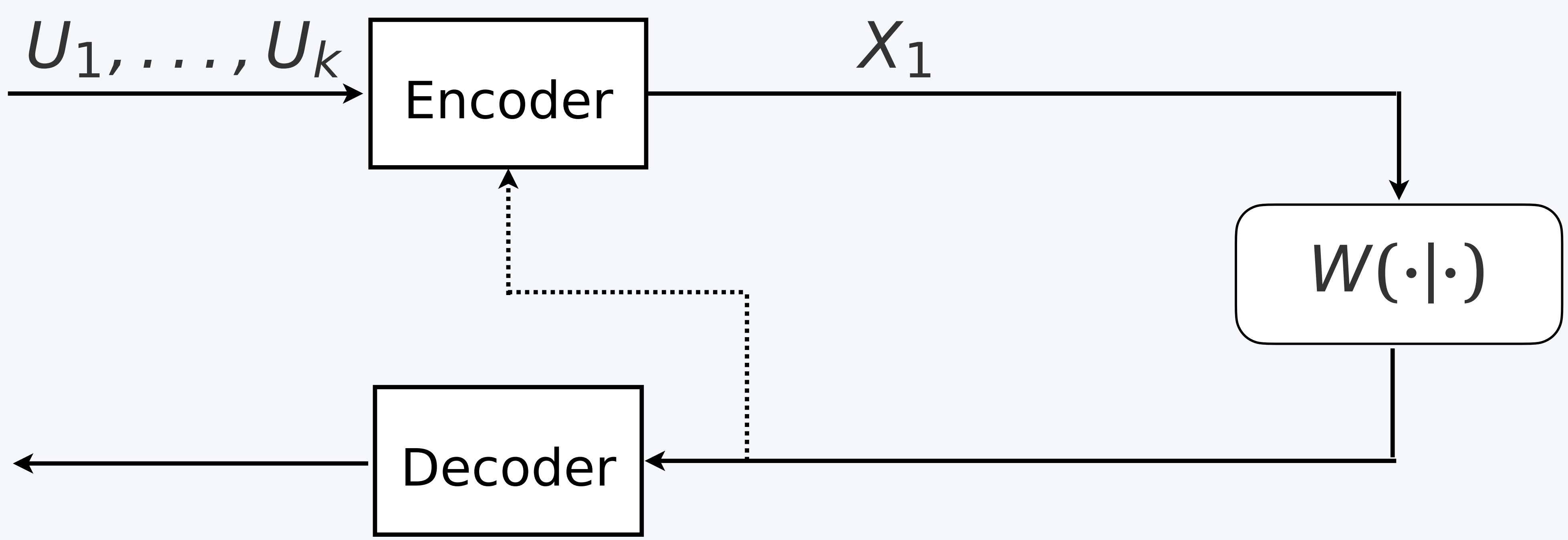

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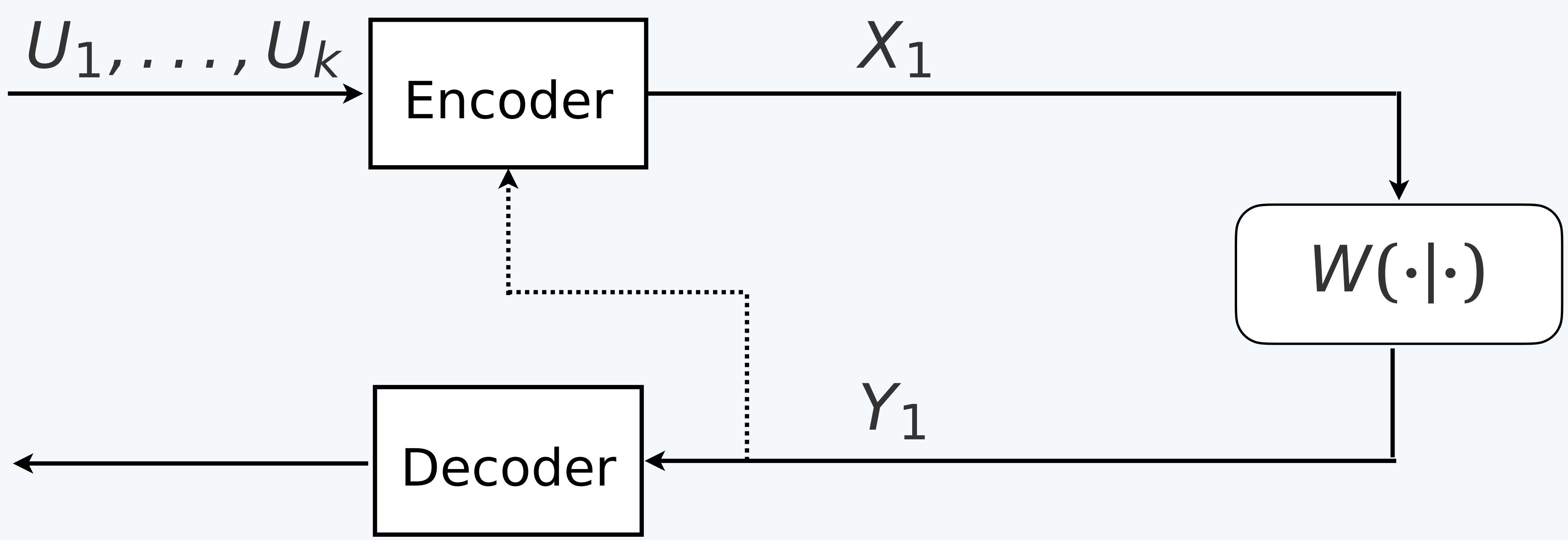

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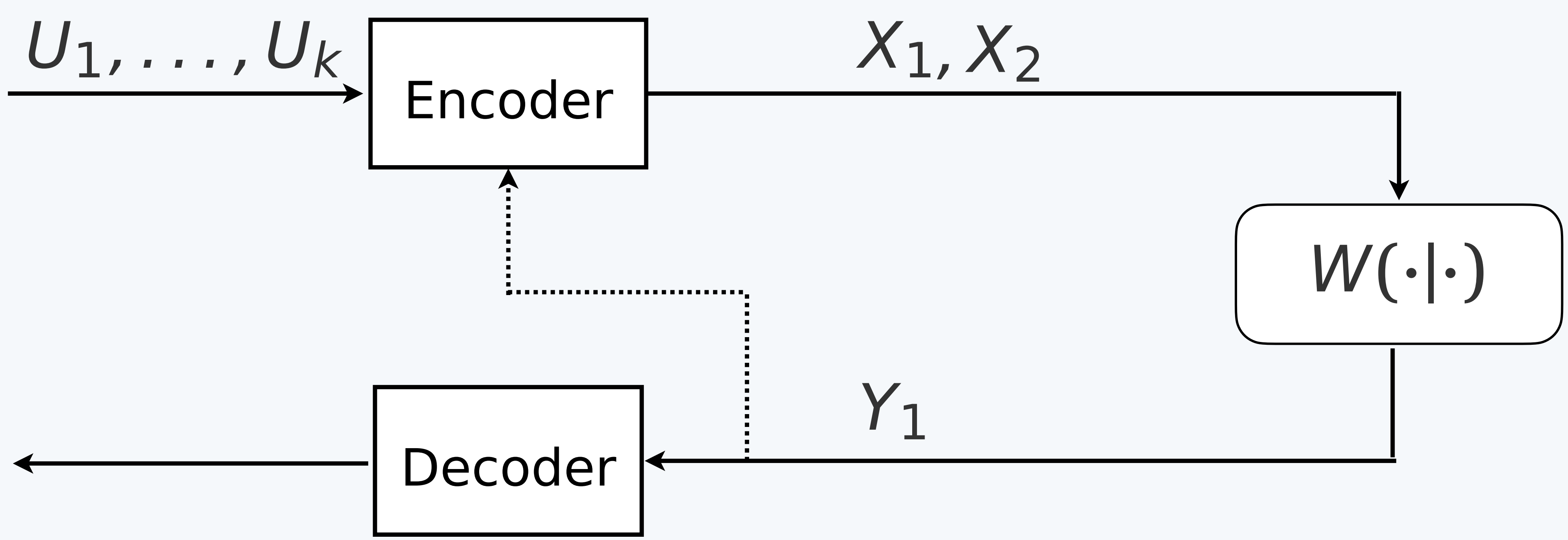

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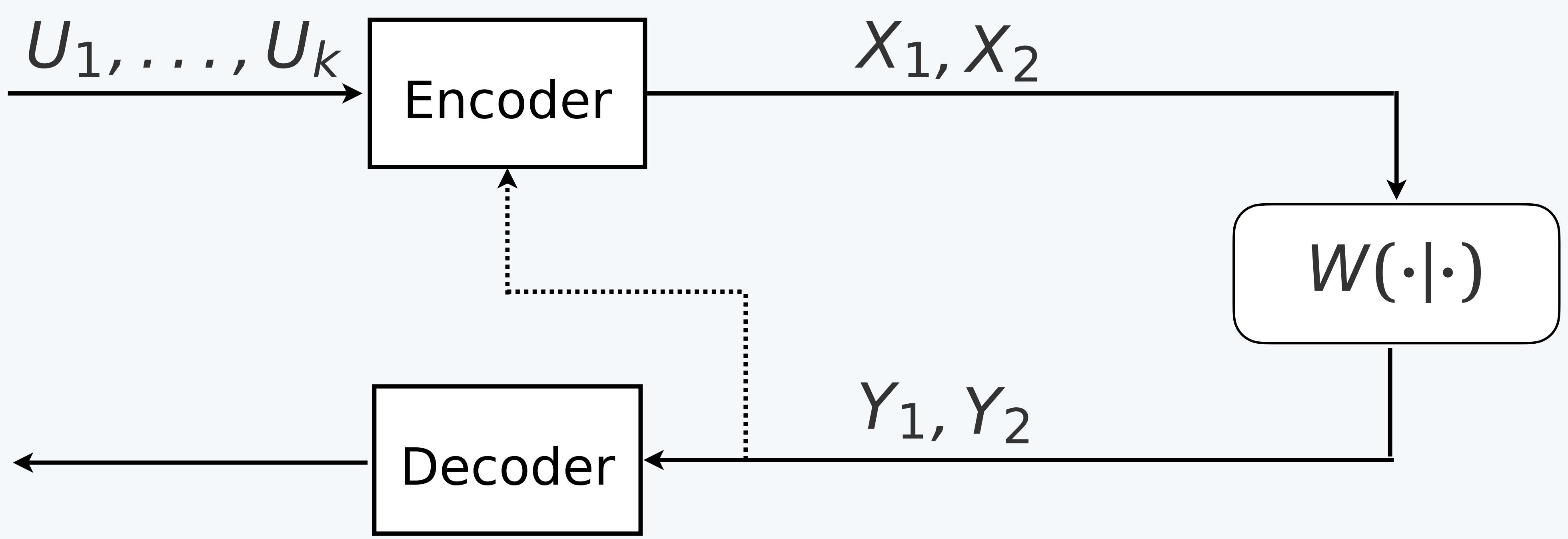

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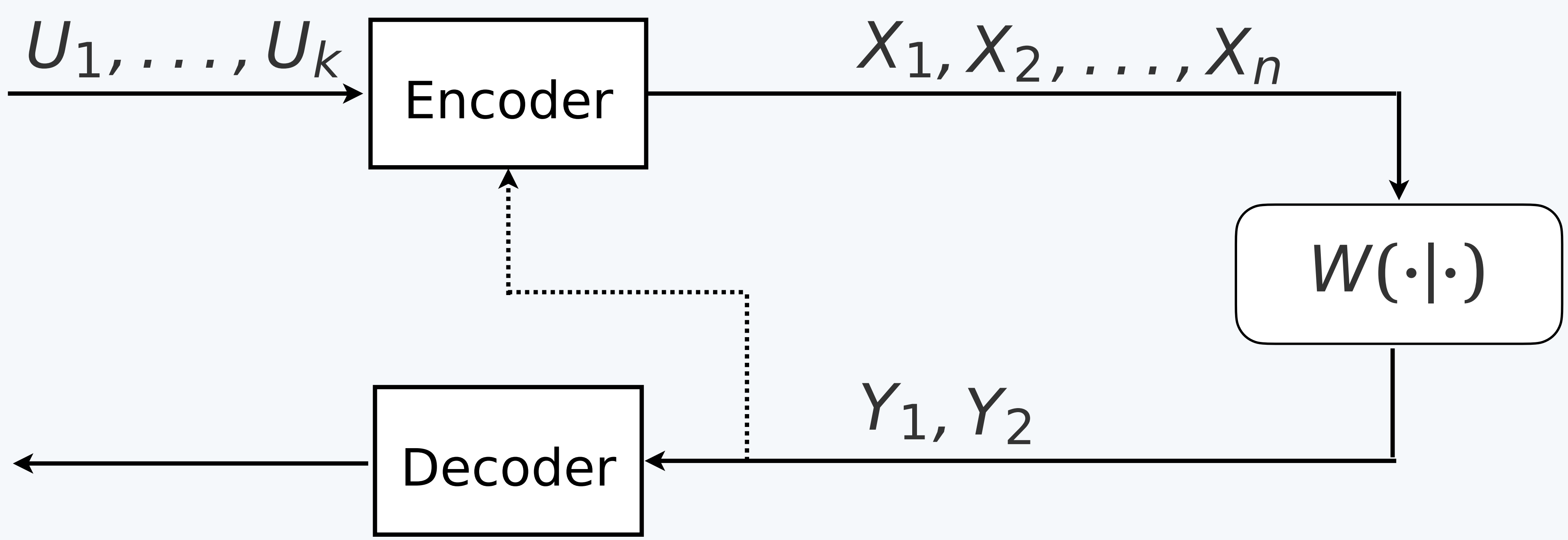

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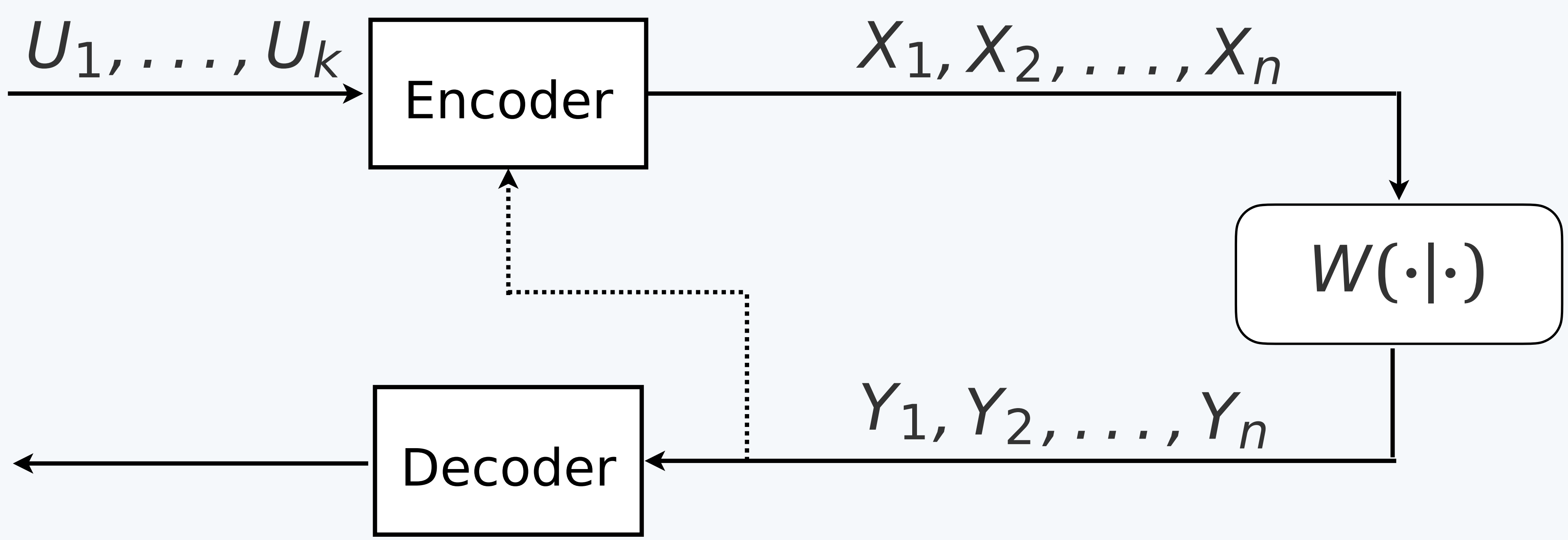

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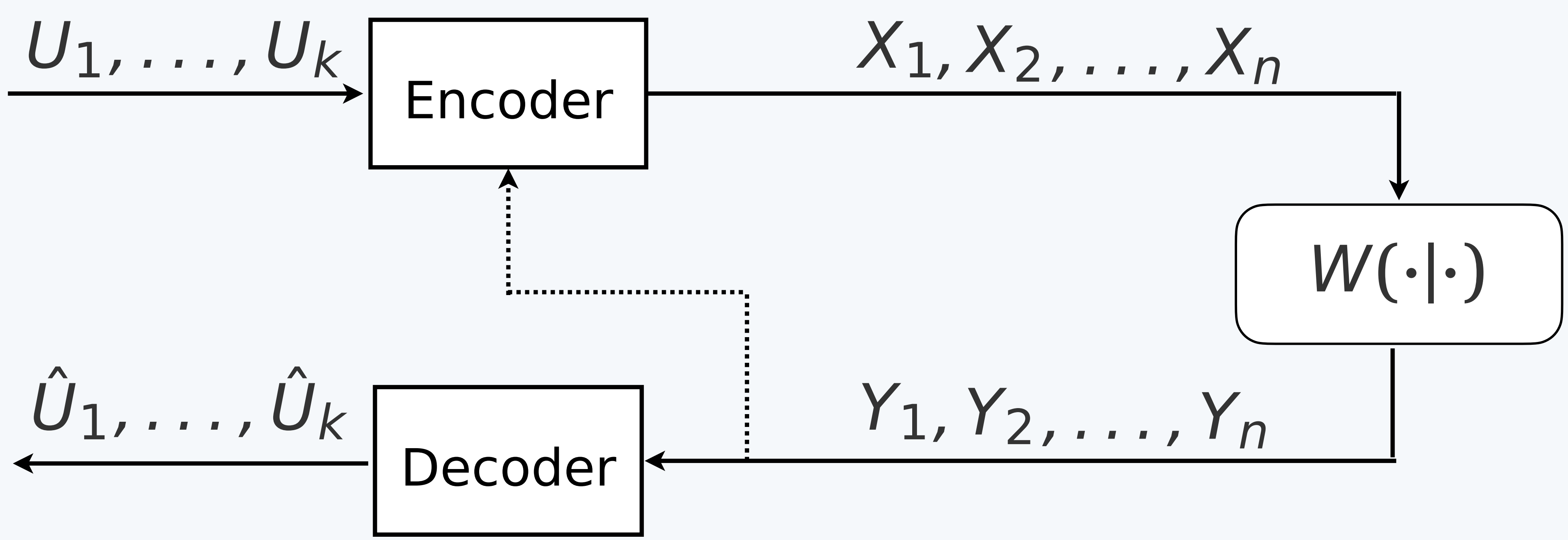

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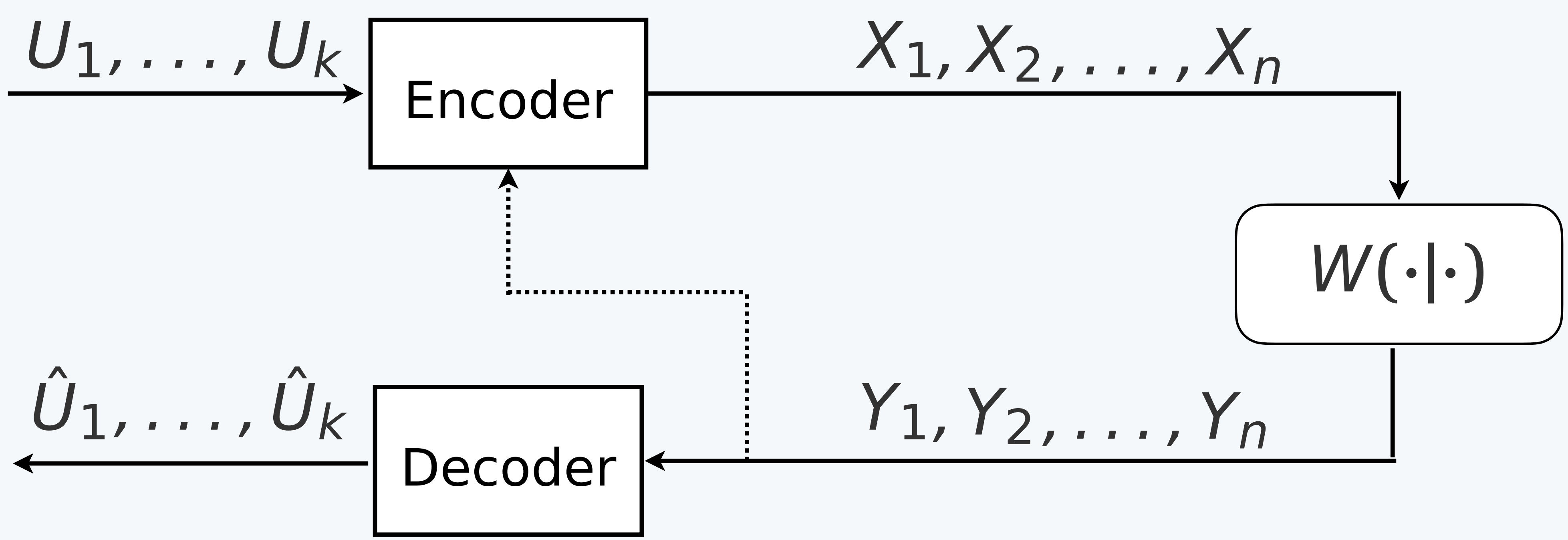
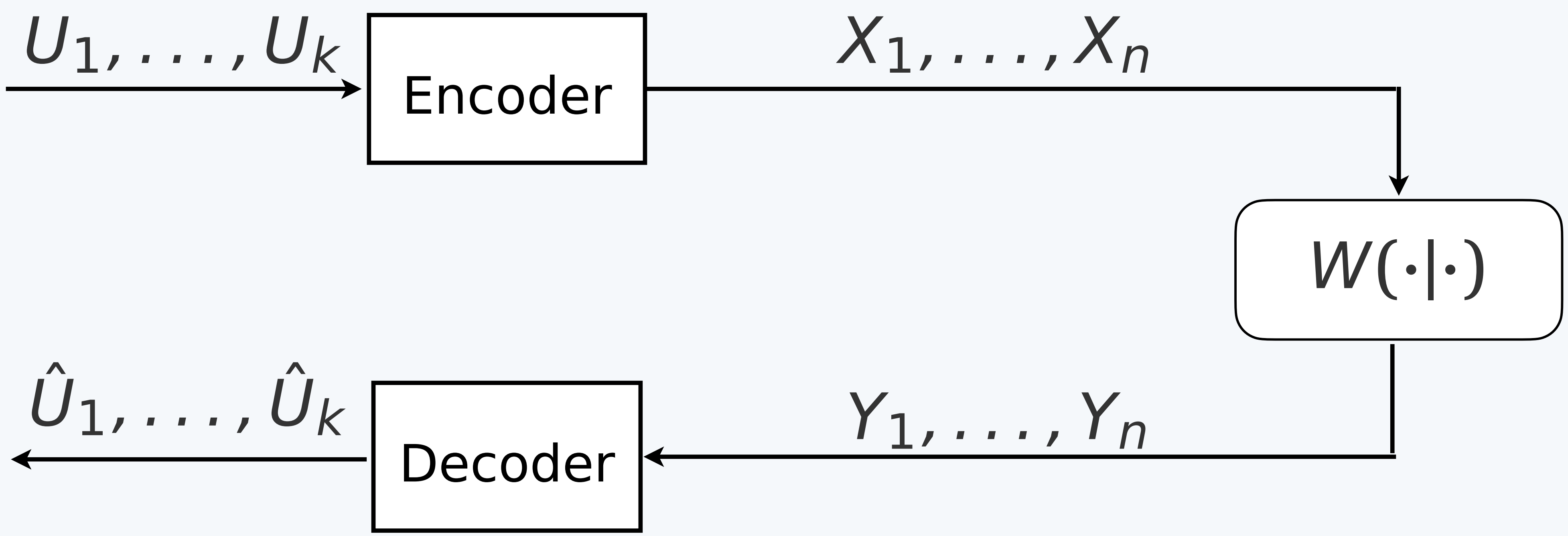


[Real feedback is never so ideal ....]

Figures of Merit



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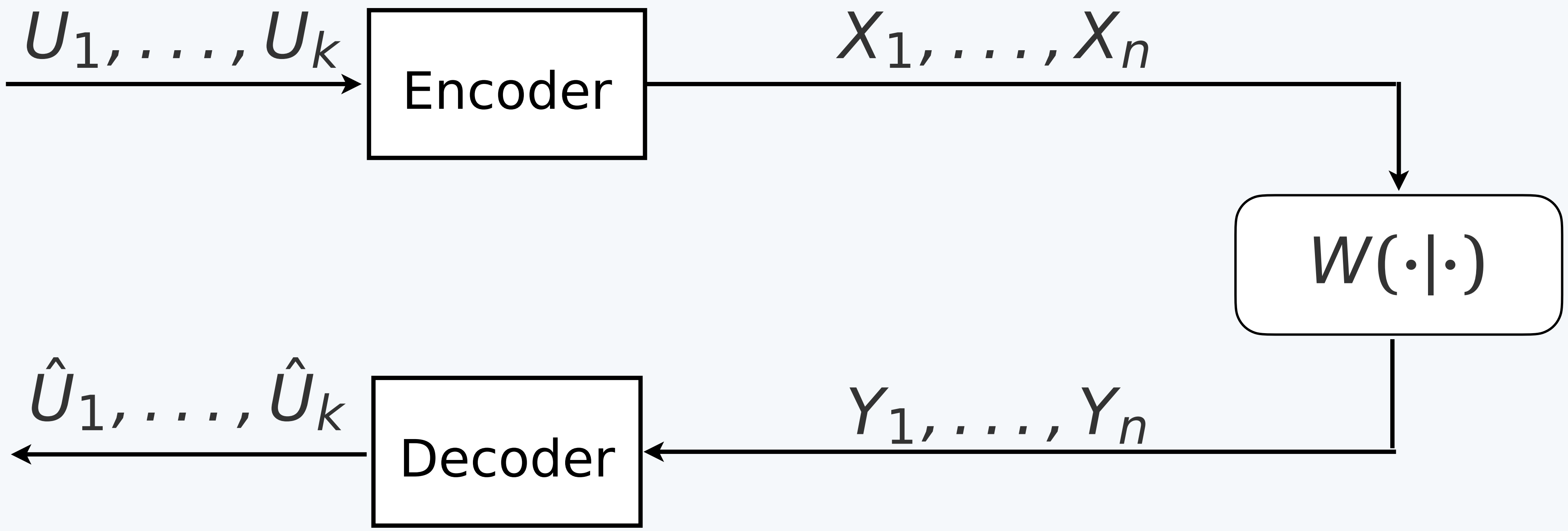

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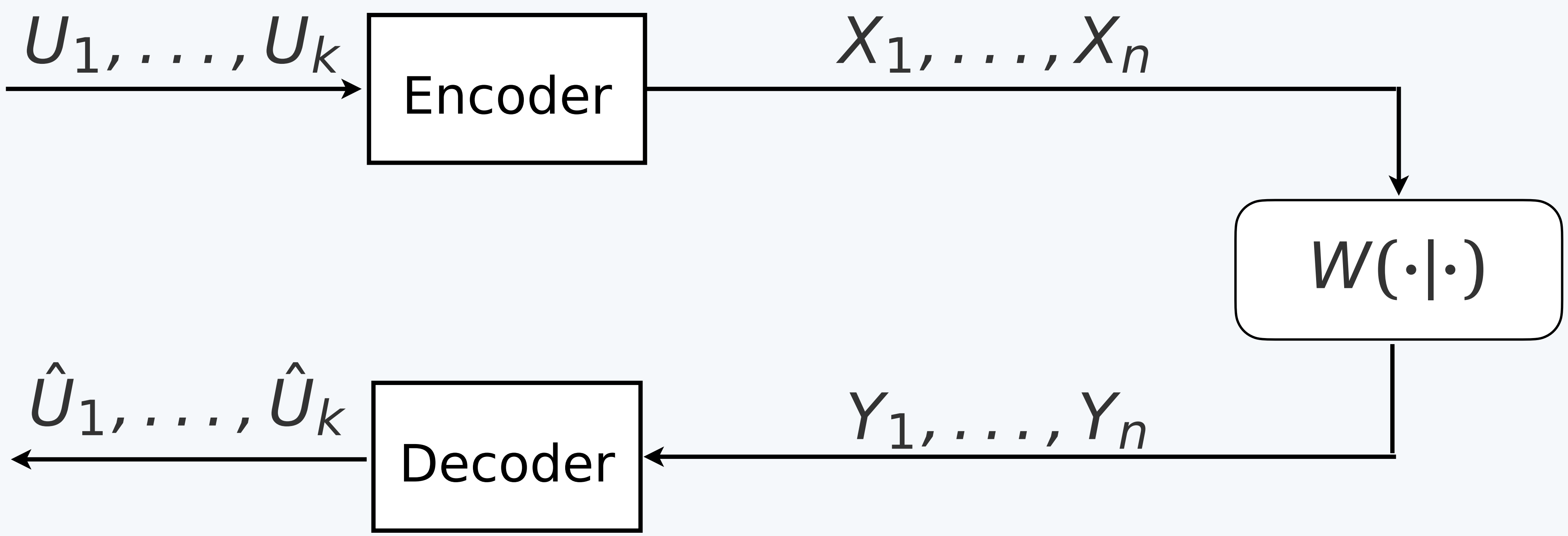

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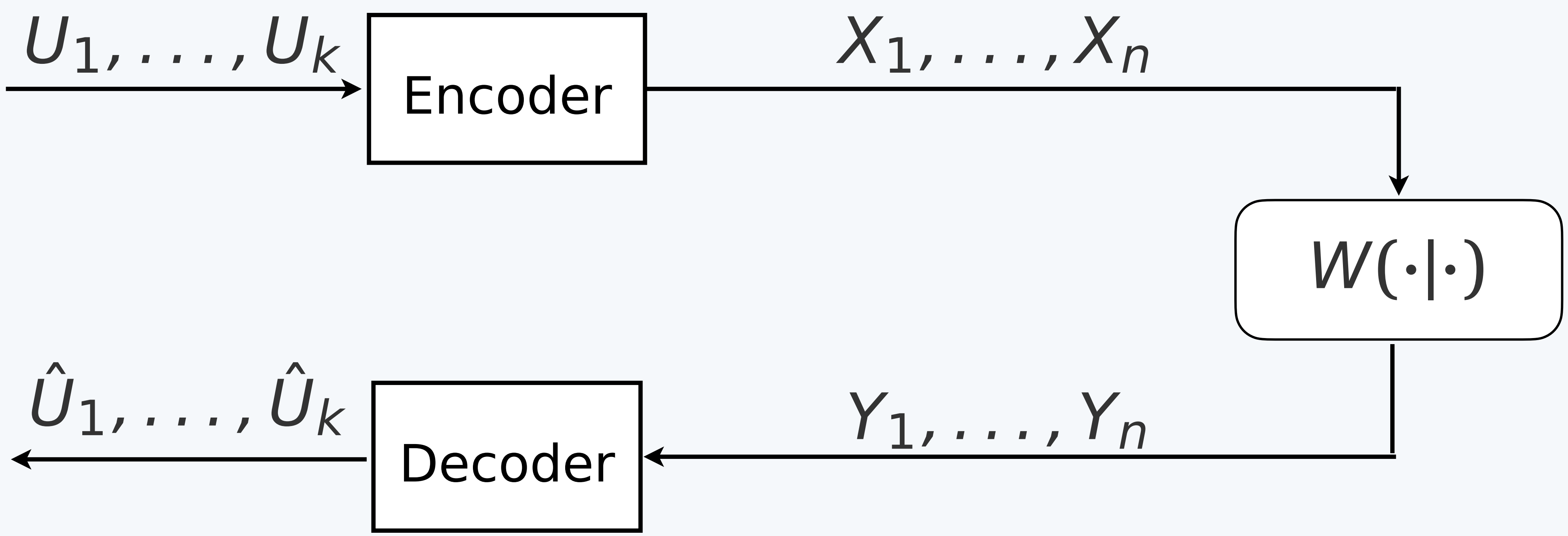

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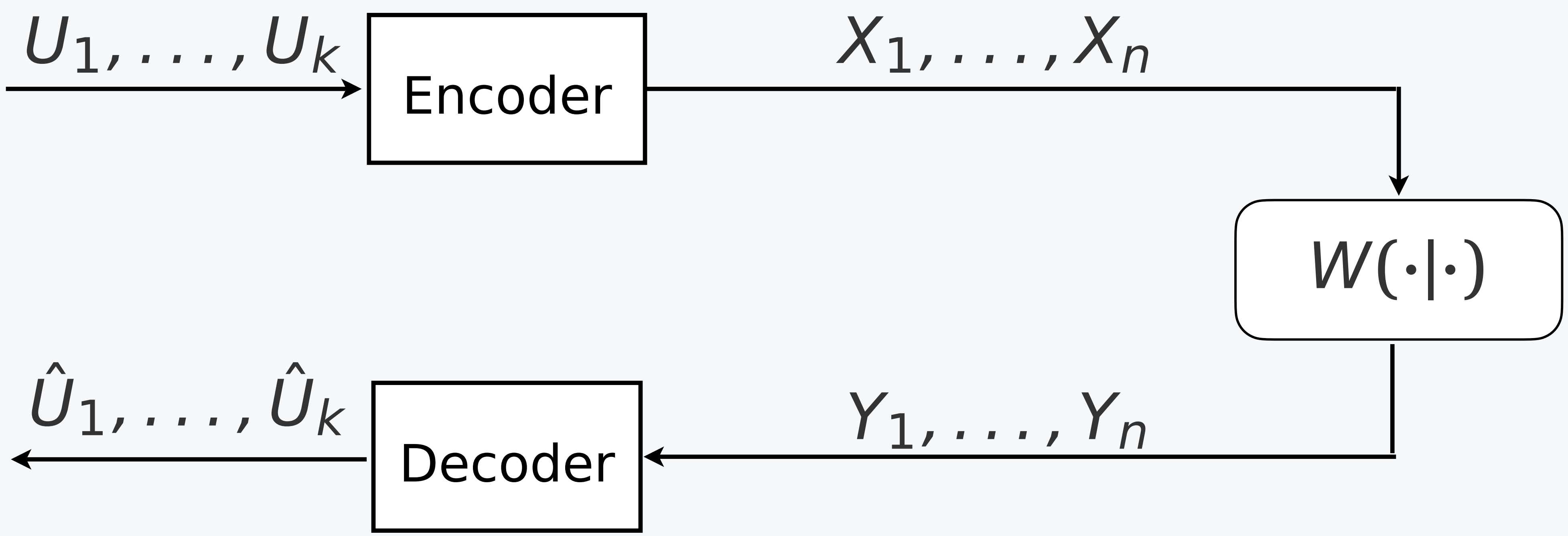

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- Error probability: $P_e = P(U^k \neq \hat{U}^k)$

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$$R(n, \epsilon) = \max \left\{ \frac{k}{n} : \exists \text{ an } (n, k, P_e) \text{ code with } P_e \leq \epsilon \right\}$$

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- **Theorem** (cf. Shannon '56)**:**

$$\lim_{n\to\infty} R(n,\epsilon) = \lim_{n\to\infty} R^{(fb)}(n,\epsilon) = C \quad \text{if } 0 < \epsilon < 1$$

where C is the capacity:

$$C = \max_P I(P;W) = \max_P E_{P\circ W}\left[\log\frac{W(Y|X)}{PW(Y)}\right]$$

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($\beta(\epsilon)$ earlier characterized by Strassen '62).

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- For *asymmetric* channels,
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  - We will show that the second-order coding rate can be improved by feedback via a novel mechanism.

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Two coins:

±$1

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- Does "feedback" help?

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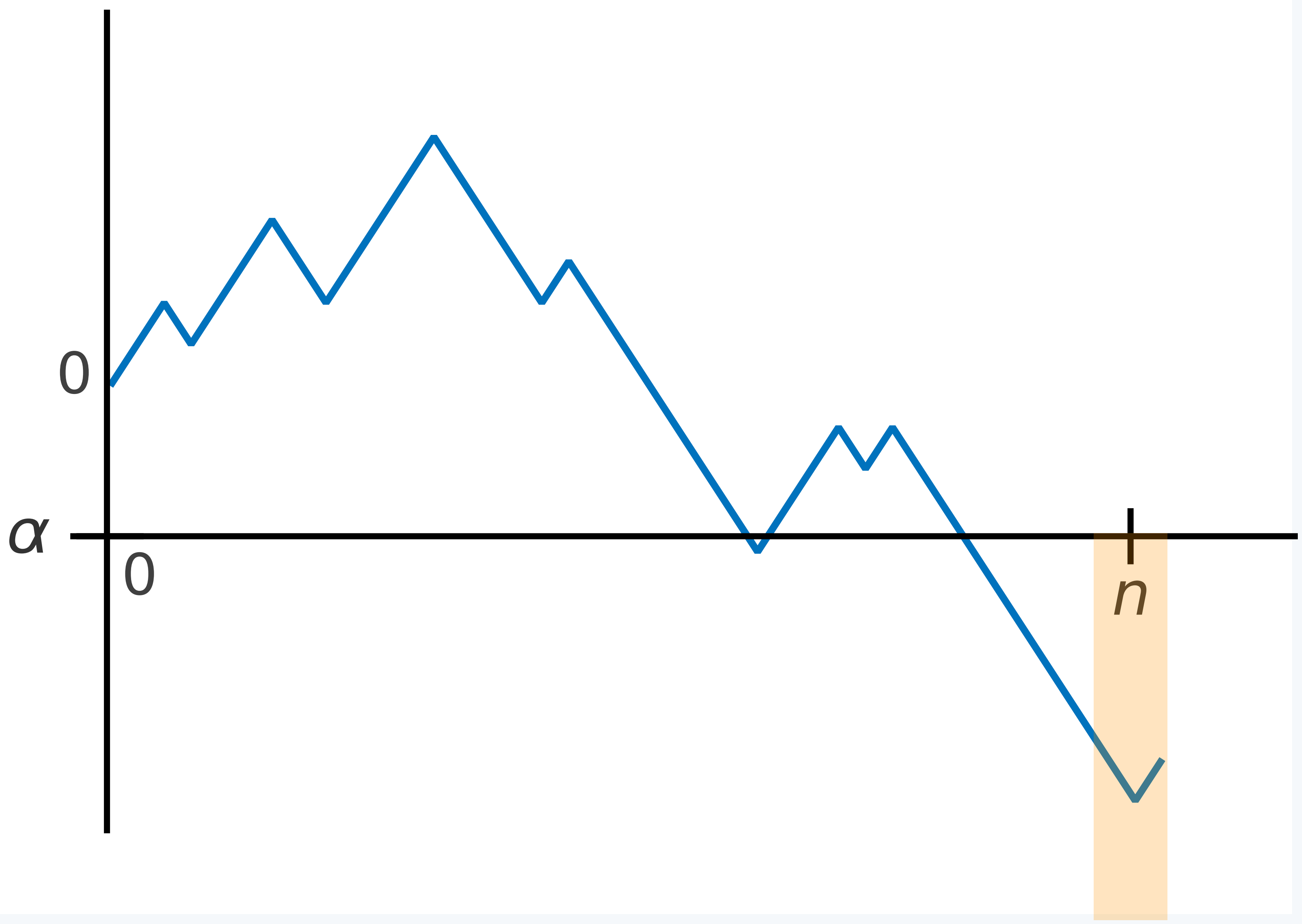
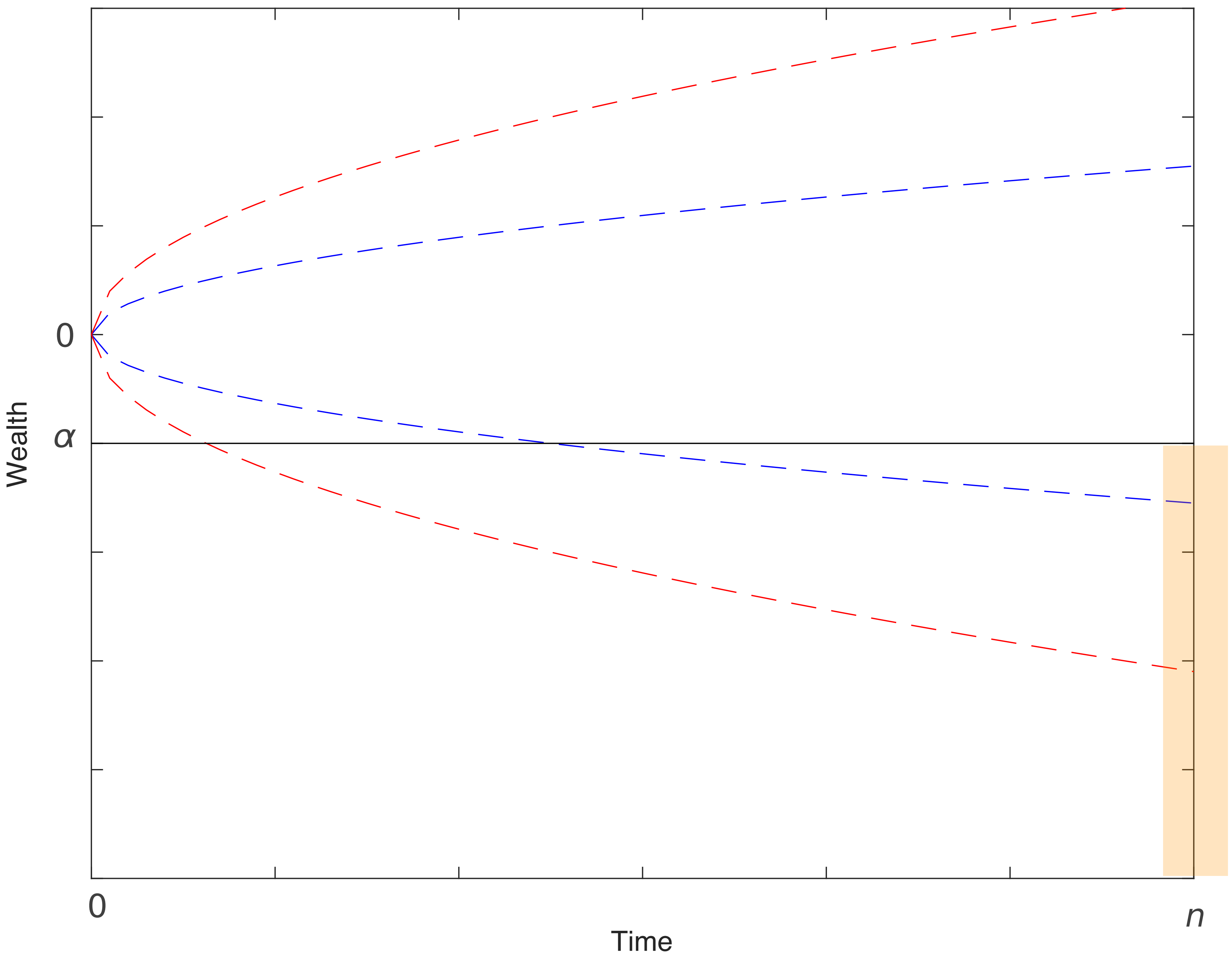


Strategies: No Feedback



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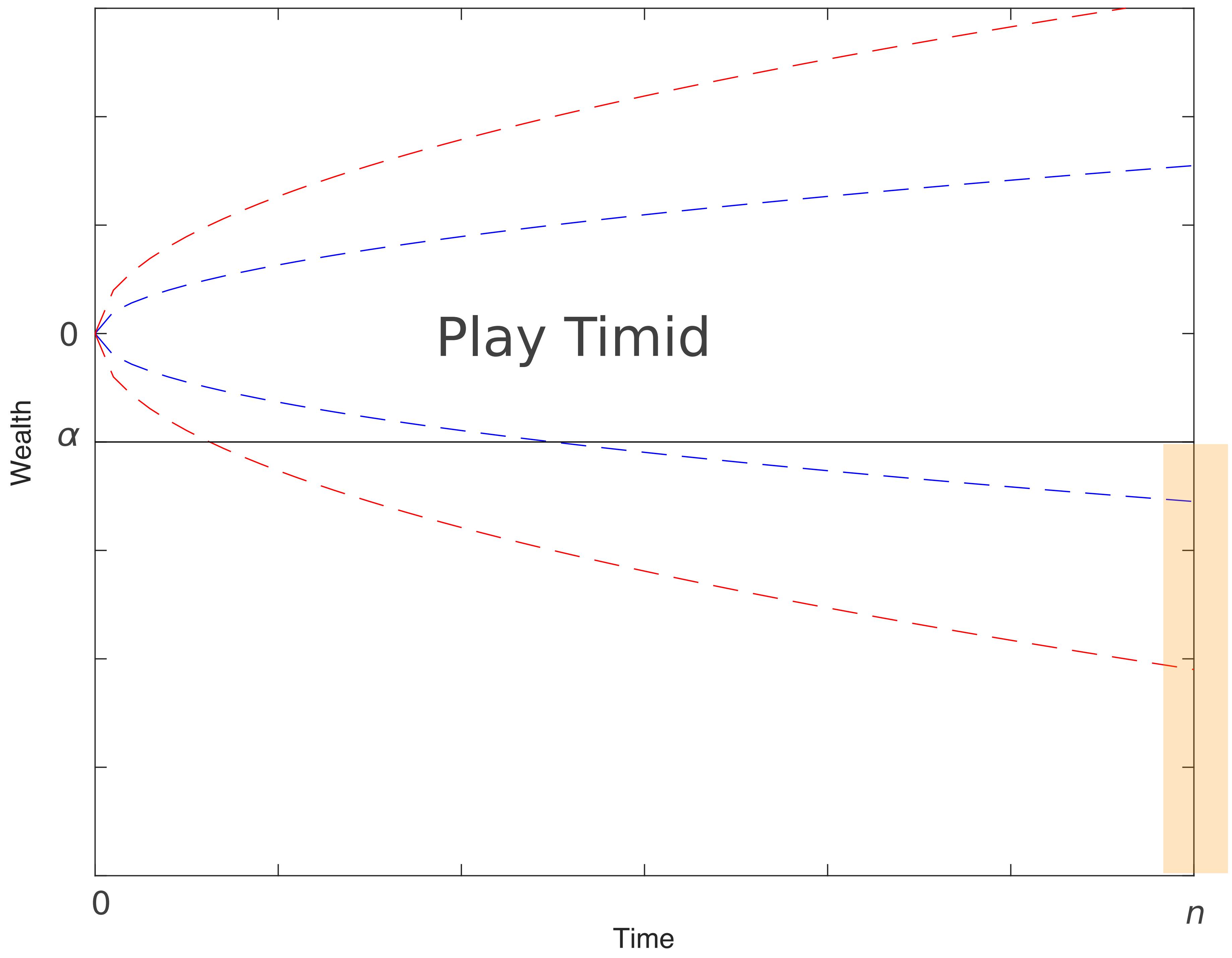

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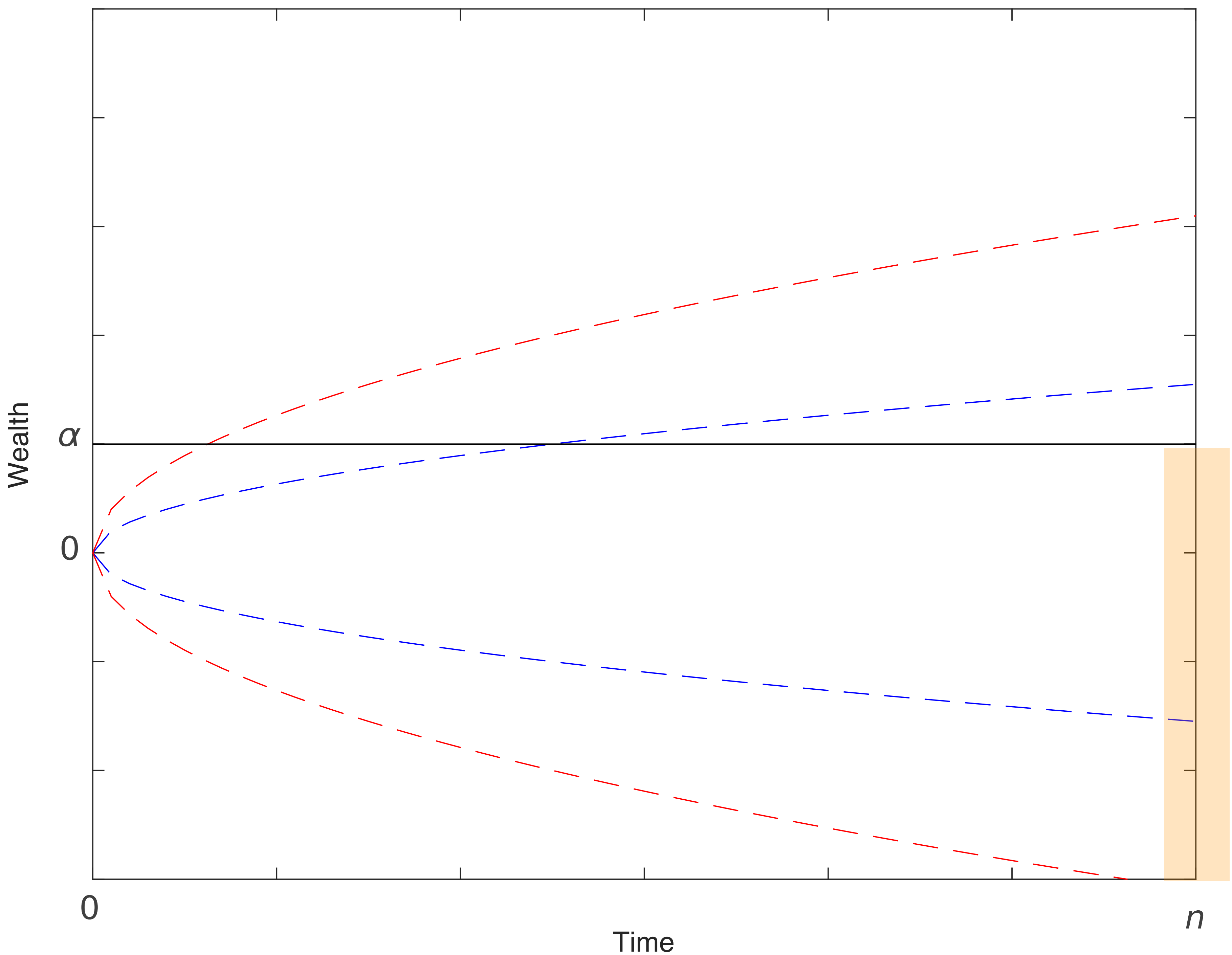

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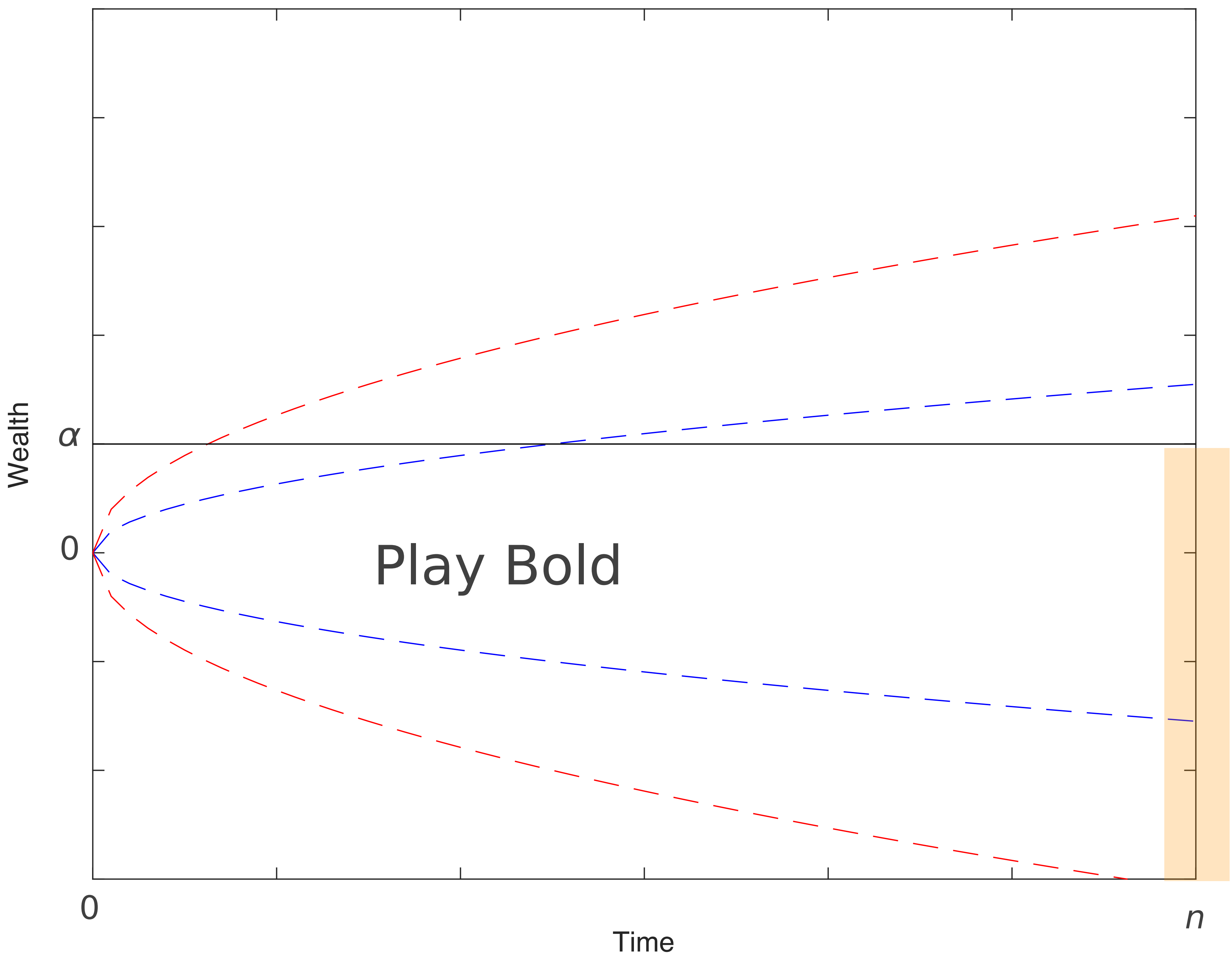

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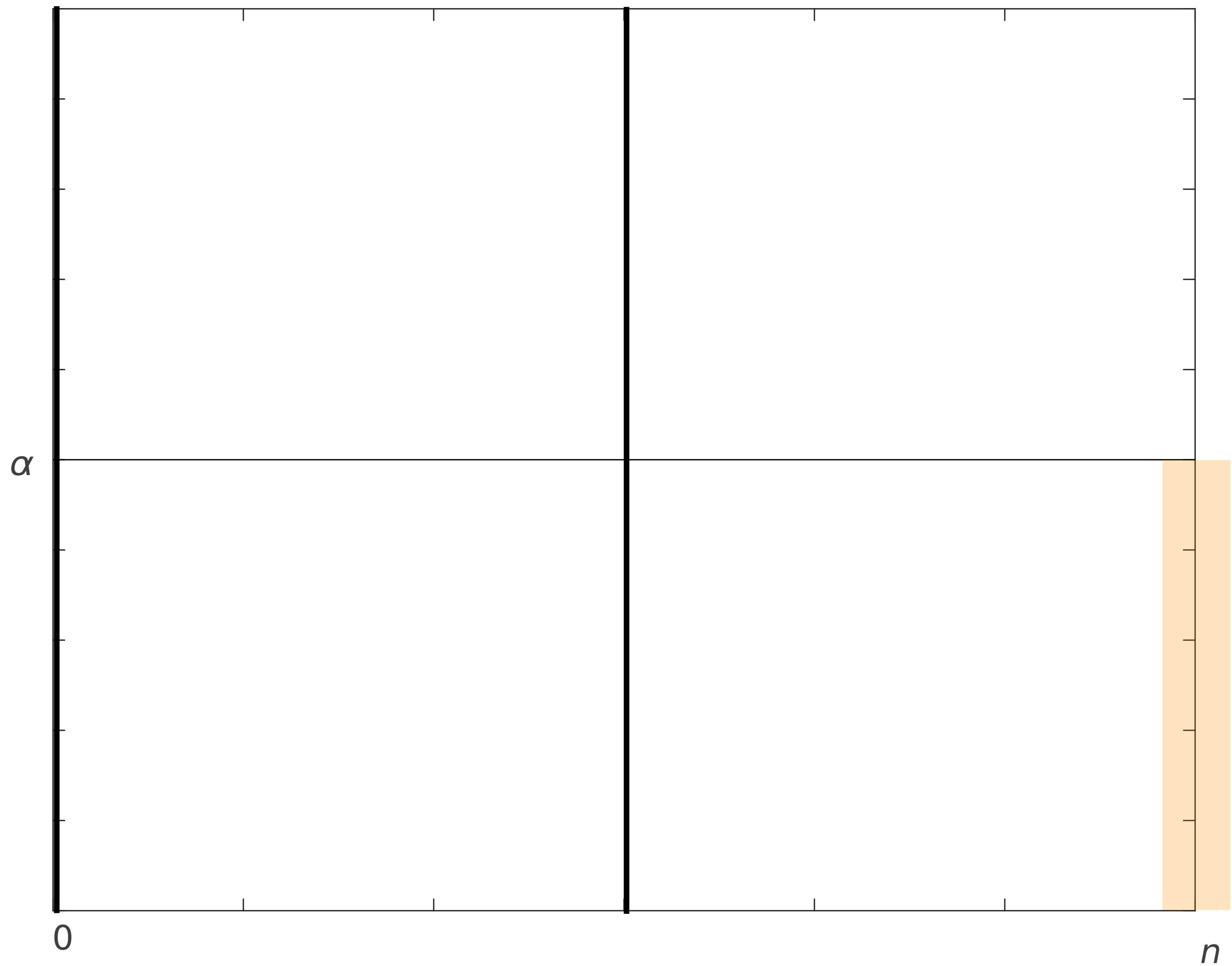

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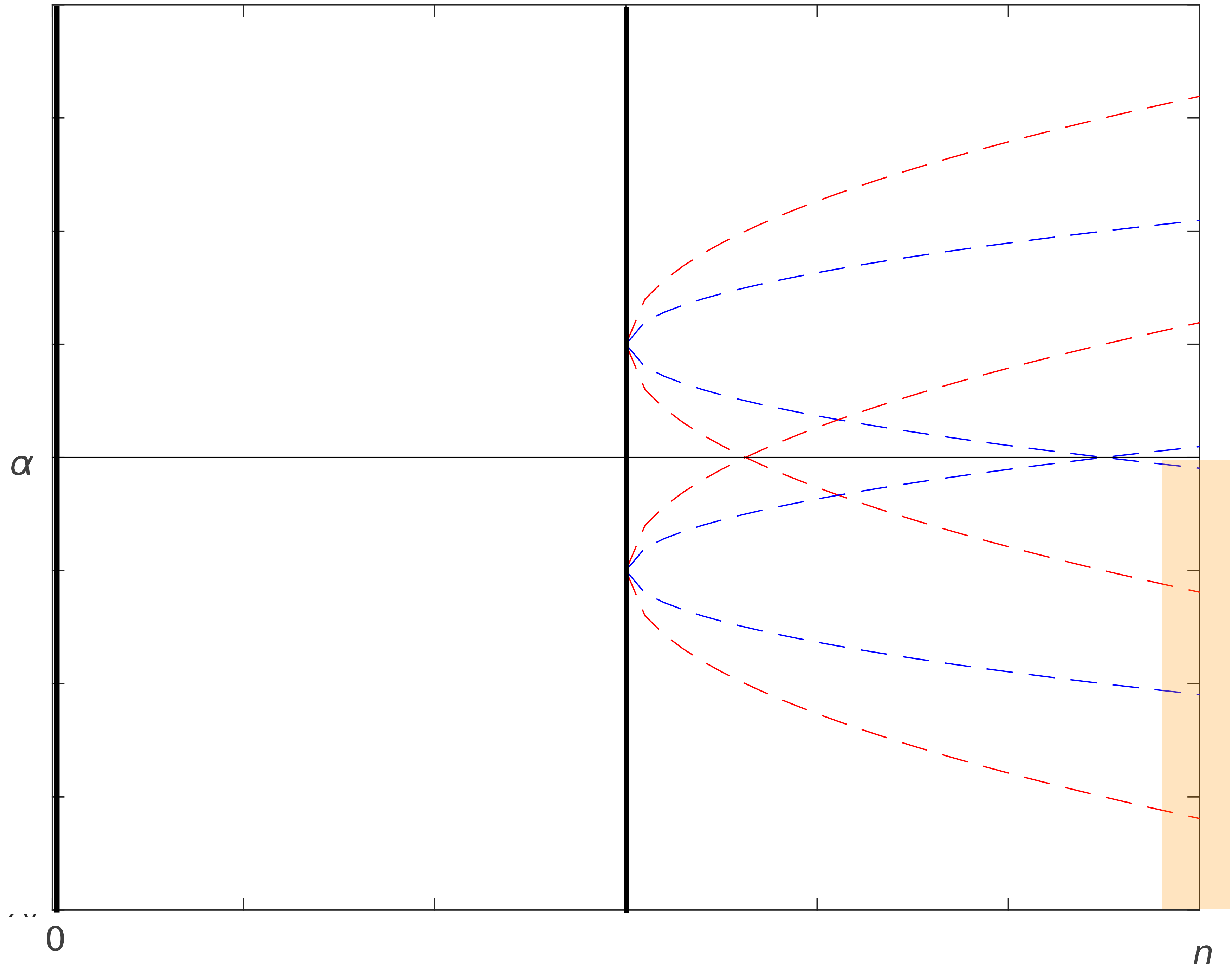

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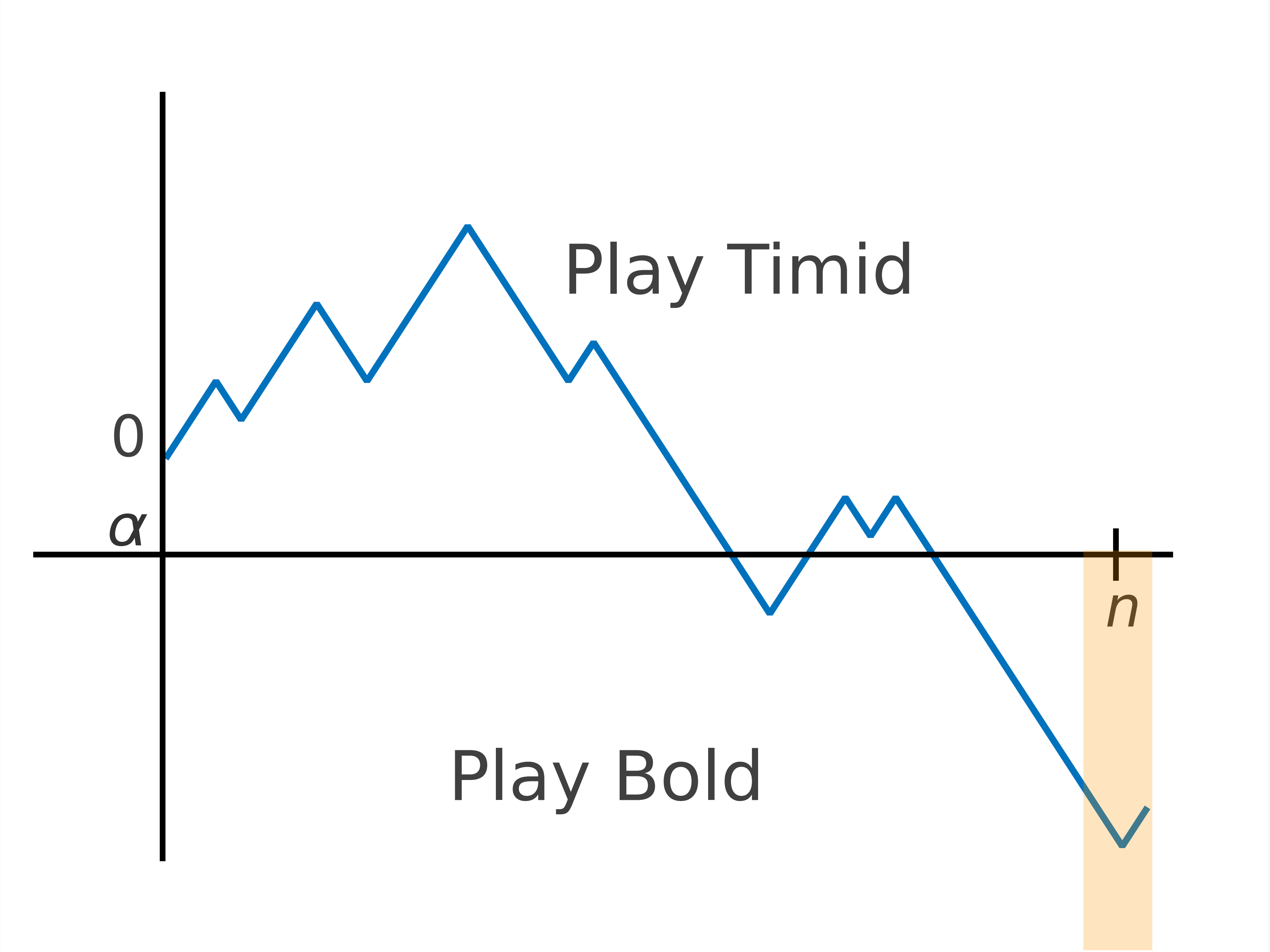

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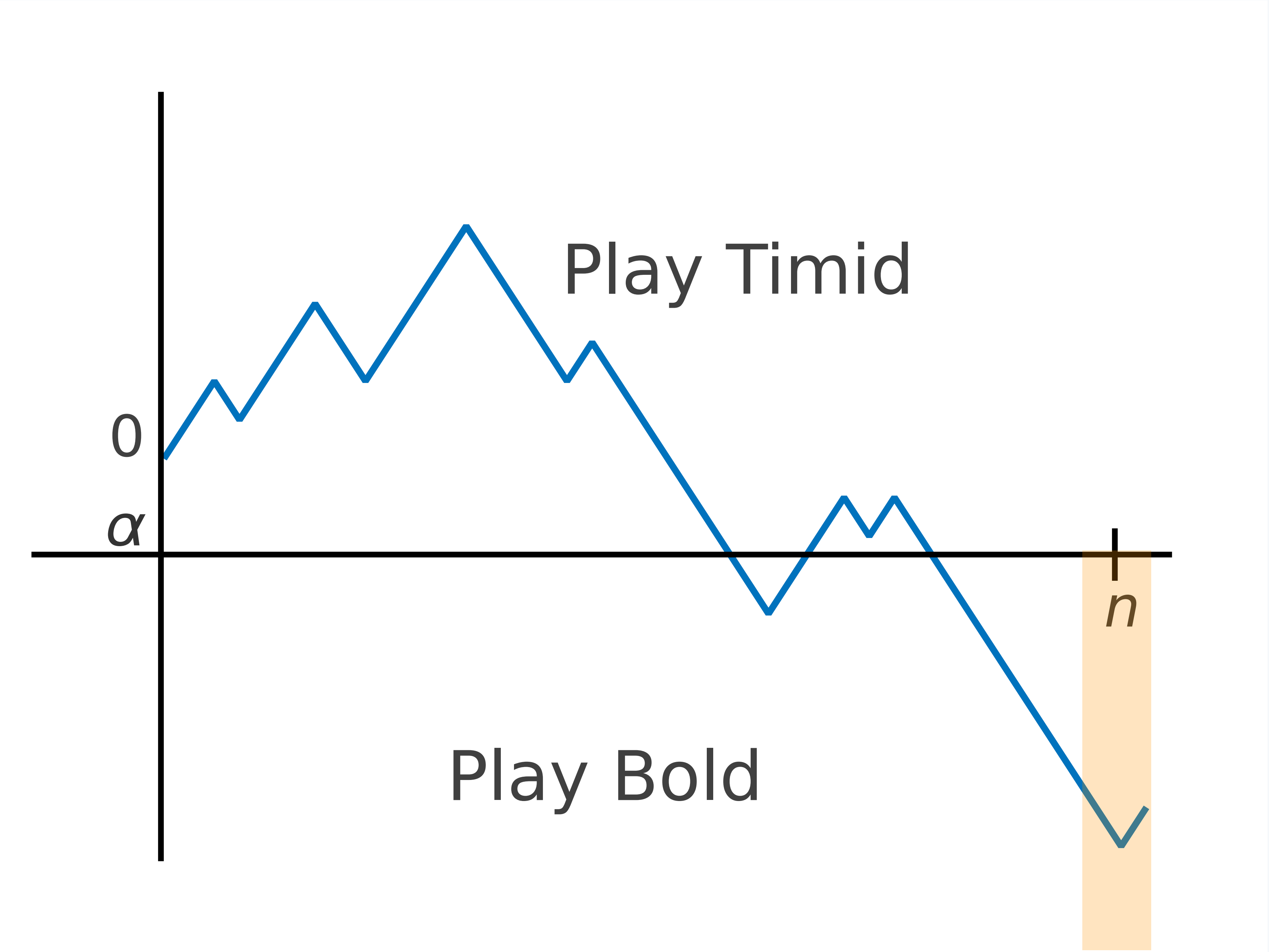

With feedback, we can do better.

On the Web

FiveThirtyEight ABC NEWS

Politics Sports Science & Health Economics **Culture**

FEB. 21, 2020, AT 8:00 AM

Can You Flip Your Way To Victory?

By Zach Wissner-Gross

Filed under The Riddler

Riddler Classic

From Abijith Krishnan comes a game of coin flipping madness:

You have two fair coins, labeled A and B. When you flip coin A, you get 1 point if it comes up heads, but you lose 1 point if it comes up tails. Coin B is worth twice as much — when you flip coin B, you get 2 points if it comes up heads, but you lose 2 points if it comes up tails.

To play the game, you make a total of 100 flips. For each flip, you can choose either coin, and you know the outcomes of all the previous flips. In order to win, you must finish with a positive total score. In your eyes, finishing with 2 points is just as good as finishing with 200 points — any positive score is a win. (By the same token, finishing with 0 or −2 points is just as bad as finishing with −200 points.)

If you optimize your strategy, what percentage of games will you win? (Remember, one game consists of 100 coin flips.)

Extra credit: What if coin A isn't fair (but coin B is still fair)? That is, if coin A comes up heads with probability *p* and you optimize your strategy, what percentage of games will you win?

Submit your answer

RECOMMENDED

Why Younger Democrats Are Overwhelmingly Rejecting Biden

Trump Approval Ratings

UPDATED 2 HOURS AGO

Strategies: Continuous-Time

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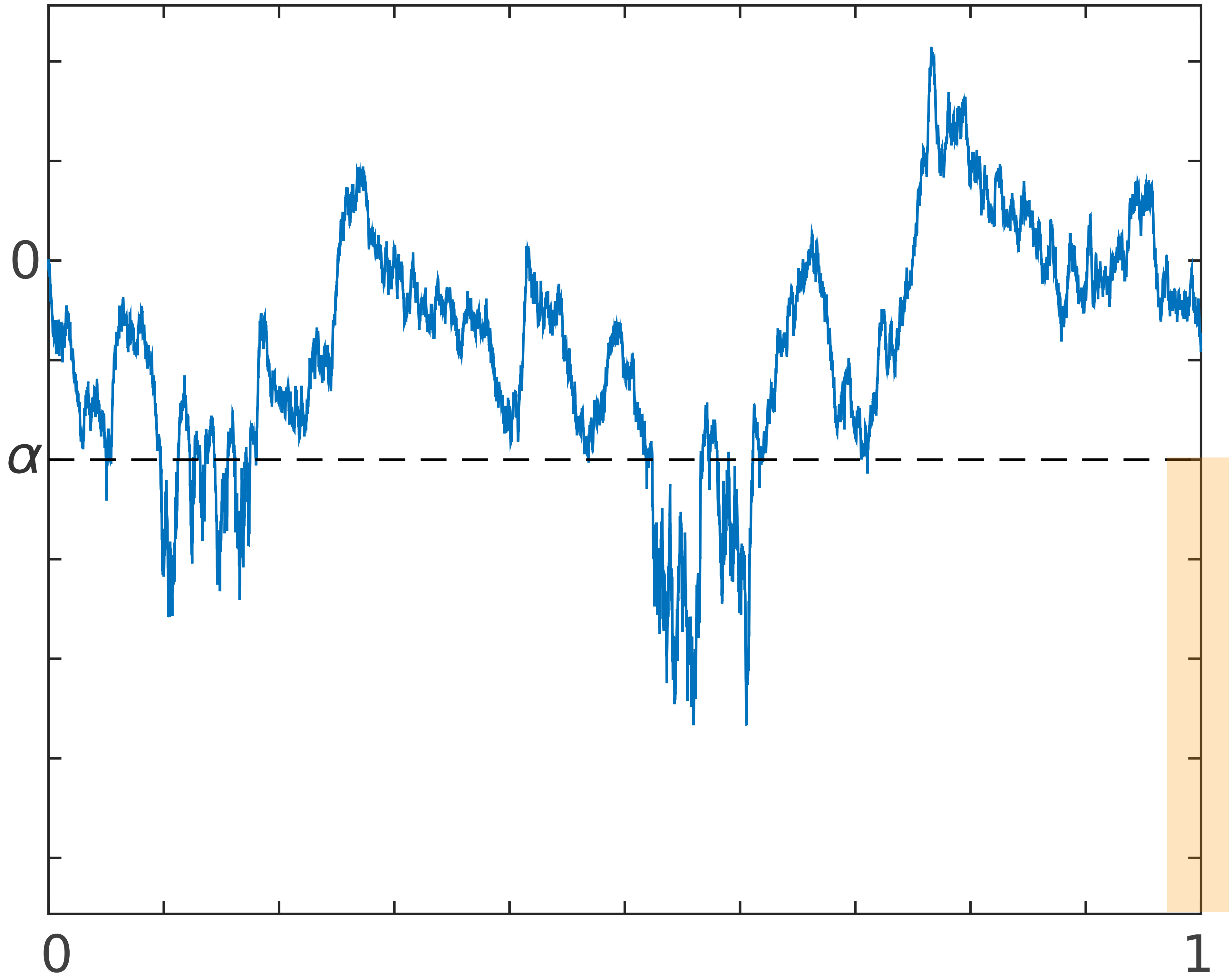
- **Theorem** (McNamara '83): The bang-bang controller

$$\sigma(x, s) = \begin{cases} \sigma_1 & \text{if } x \geq \alpha \\ \sigma_2 & \text{if } x < \alpha \end{cases}$$

  is optimal.

In Simulation



In the Real World

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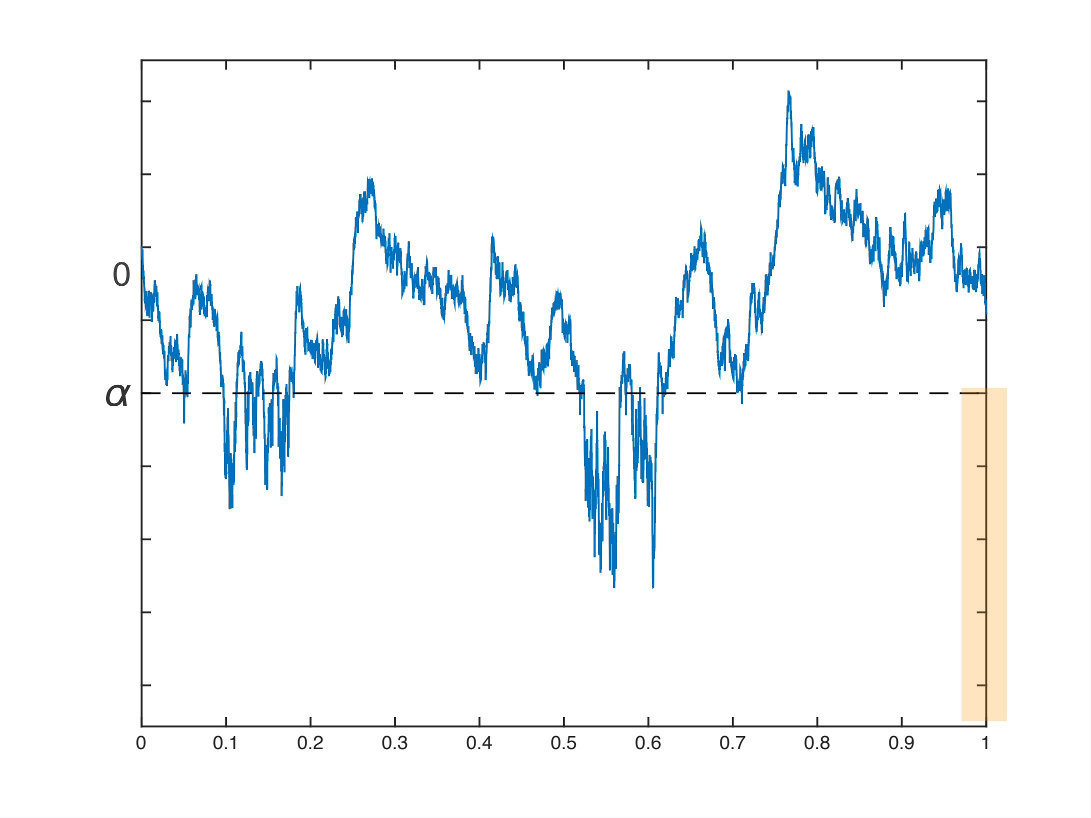
- McNamara ’83: foraging animals

The McNamara Threshold



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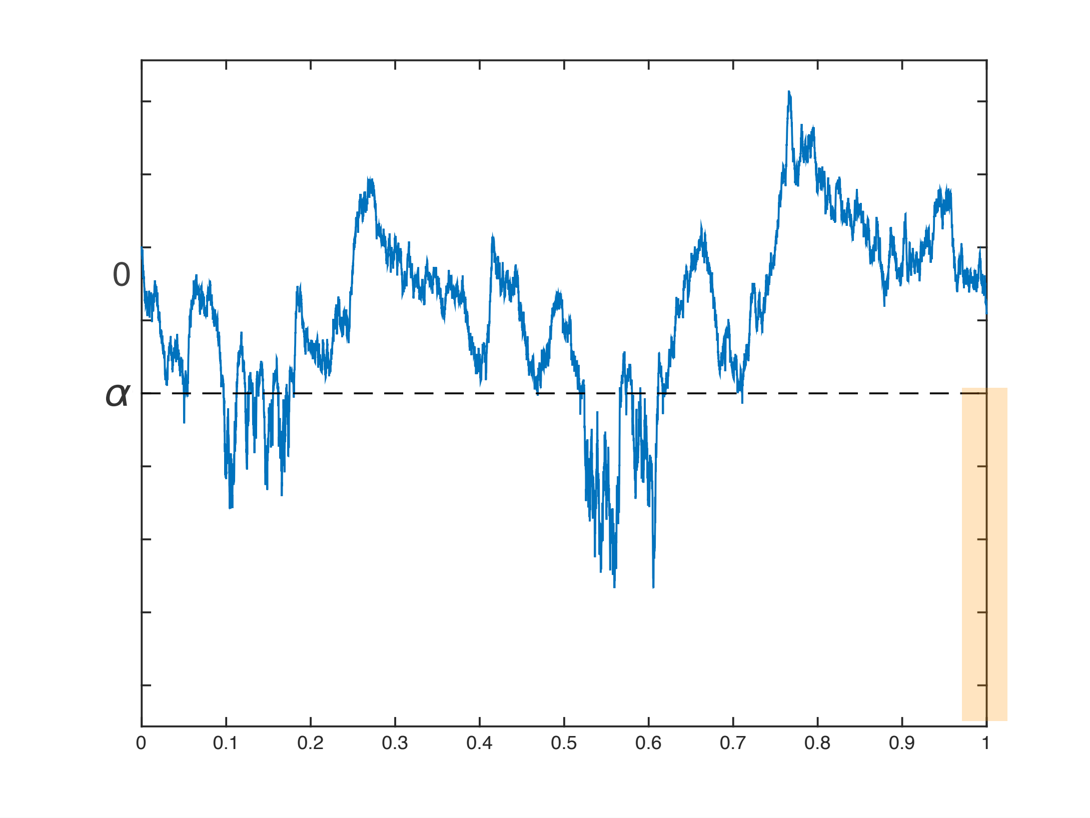

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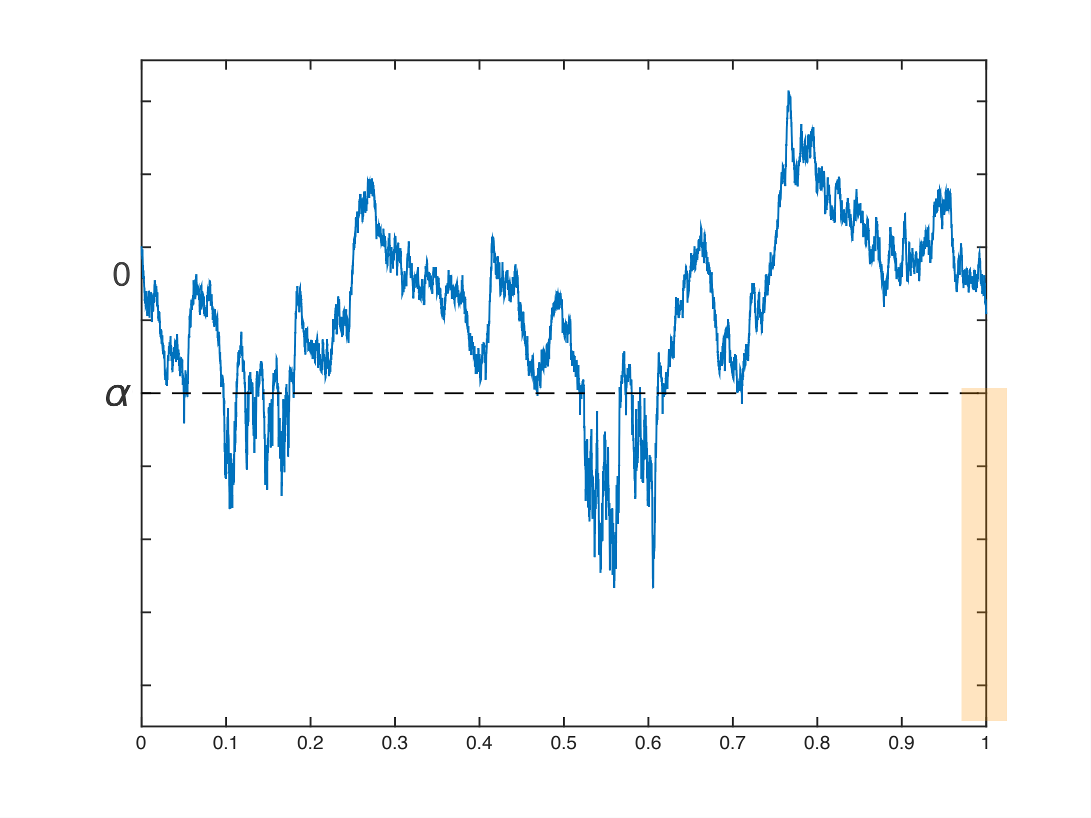

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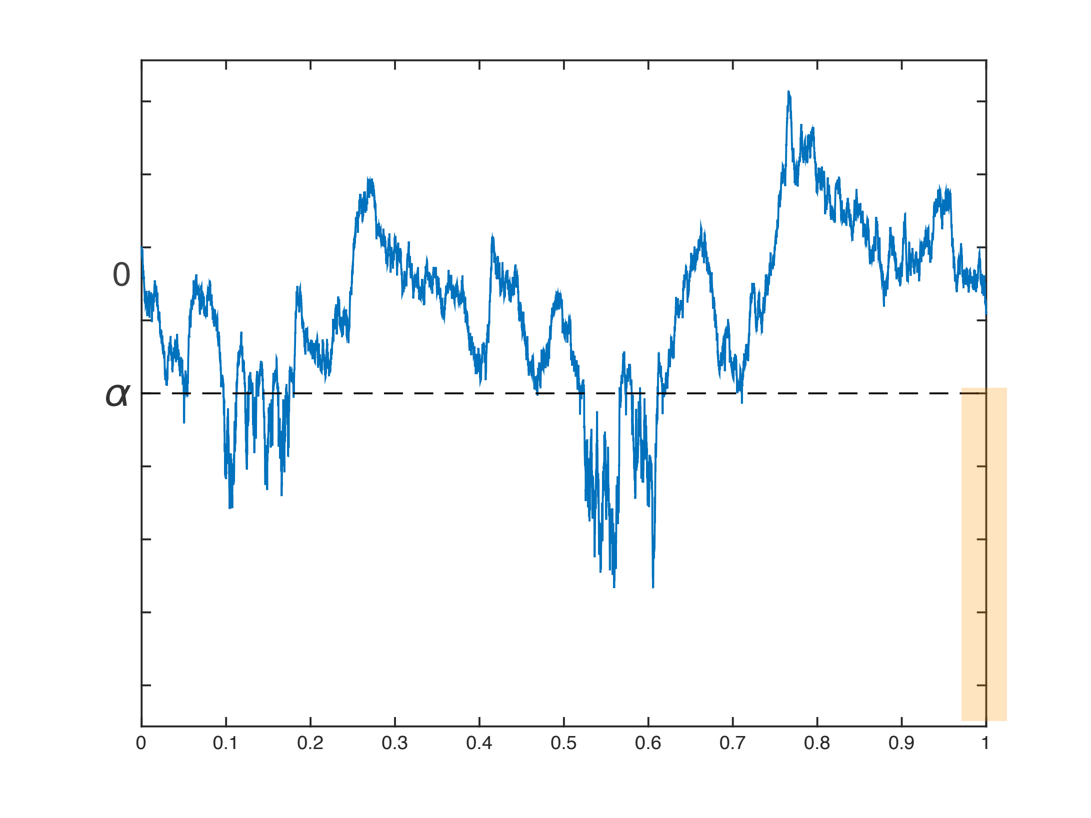

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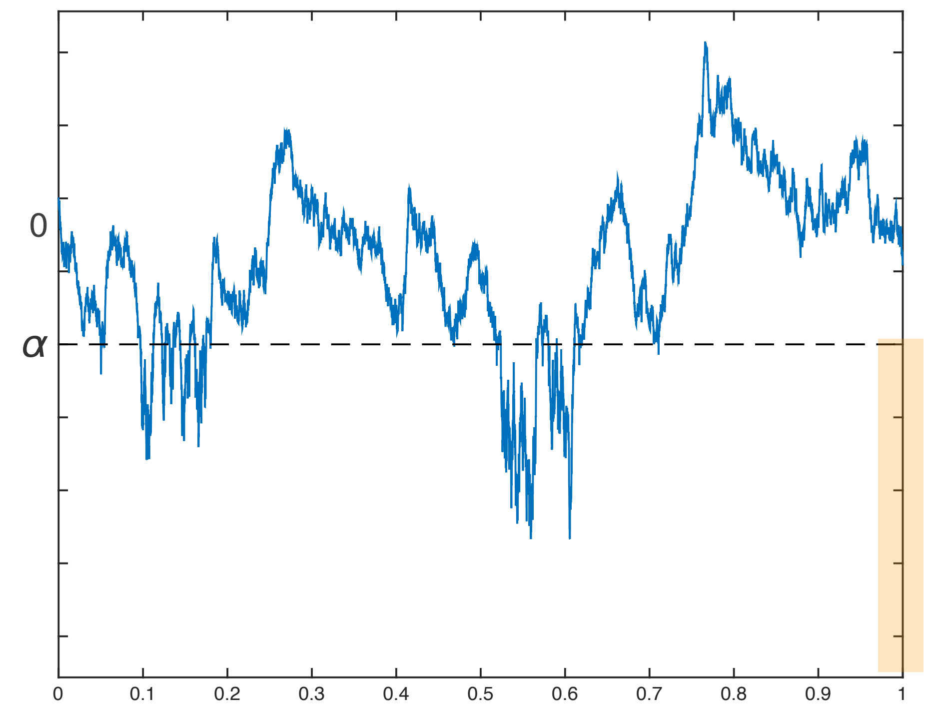

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- Both expressible in terms of inv. Gaussian CDF
- **Lemma**: $\Gamma(\epsilon, \sigma_1, \sigma_2) < \Gamma^{(fb)}(\epsilon, \sigma_1, \sigma_2)$ iff $\sigma_1 < \sigma_2$

A Key Lemma

- **Def**: a *controller* is a function

$$f : (\mathcal{X} \times \mathcal{Y})^* \to \mathcal{P}(\mathcal{X})$$

which along with the channel W defines a joint distribution

$$(f \circ W)(x^n, y^n) = \prod_{i=1}^{n} (f(x^{i-1}, y^{i-1}))(x_i) W(y_i | x_i)$$

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- \$64K question: can we control the variance of the increments?

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Compound dispersion if $V_{\min} < V_{\max}$. Otherwise *simple dispersion.*

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A channel with a unique capacity-achieving input distribution is necessarily simple dispersion.

A Compound Dispersion Example

$$W(y|x) = \begin{bmatrix} p & 0.5(1-p) & 0.5(1-p) \\ 0.5(1-p) & p & 0.5(1-p) \\ 0.5(1-p) & 0.5(1-p) & p \\ \hline q & 1-q & 0 \\ 0 & q & 1-q \\ 1-q & 0 & q \end{bmatrix}$$

$$\begin{array}{lr} \text{if} & p = 0.8 \\ \text{and} & q \approx 0.337 \\ \text{then} & V_{\text{min}} = .102 \\ & V_{\text{max}} = .692 \end{array}$$

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[Many more examples when we consider cost constraints ...]

SOCR without Feedback

- **Theorem 0:** (Strassen '62) For any DMC, the SOCR satisfies:

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- **Intuition:** By the key lemma, SOCR is the max α such that

$$\lim_{n\to\infty} \inf_f (f \circ W)\left(\sum_{i=1}^{n}\left(\log\frac{W(Y_i|X_i)}{(fW)(Y_i|Y^{i-1})} - C\right) \le \alpha\sqrt{n}\right) < \epsilon$$

where f is "open-loop." Intuitively, optimal choice *should* be:

$$\begin{cases} P_{\mathrm{min}} & \text{if } \Gamma(\epsilon, \sqrt{V_{\mathrm{min}}}, \sqrt{V_{\mathrm{max}}}) < 0 \\ P_{\mathrm{max}} & \text{if } \Gamma(\epsilon, \sqrt{V_{\mathrm{min}}}, \sqrt{V_{\mathrm{max}}}) > 0 \end{cases}$$

Second-Order Coding Rate w/ Feedback

- **Theorem 1** (Wagner-Shende-Altuğ '20): For any DMC with feedback,

$$\begin{aligned}\beta^{(fb)}(\epsilon) &\geq \Gamma^{(fb)}\left(\epsilon, \sqrt{V_{\min}}, \sqrt{V_{\max}}\right) \\ &> \Gamma\left(\epsilon, \sqrt{V_{\min}}, \sqrt{V_{\max}}\right) \text{ if } V_{\max} > V_{\min} \\ &= \beta(\epsilon)\end{aligned}$$

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- **Corollary** (Wagner-Shende-Altuğ '20): Feedback improves the second-order coding rate for any compound-dispersion DMC.

Proof of Theorem 1

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- Key lemma:

$$\lim_{n\to\infty} \inf_f (f \circ W)\left(\sum_{i=1}^{n}\left(\log \frac{W(Y_i|X_i)}{(fW)(Y_i|Y^{i-1})} - C\right) \leq \alpha\sqrt{n}\right) < \epsilon$$

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Timid/Bold Coding

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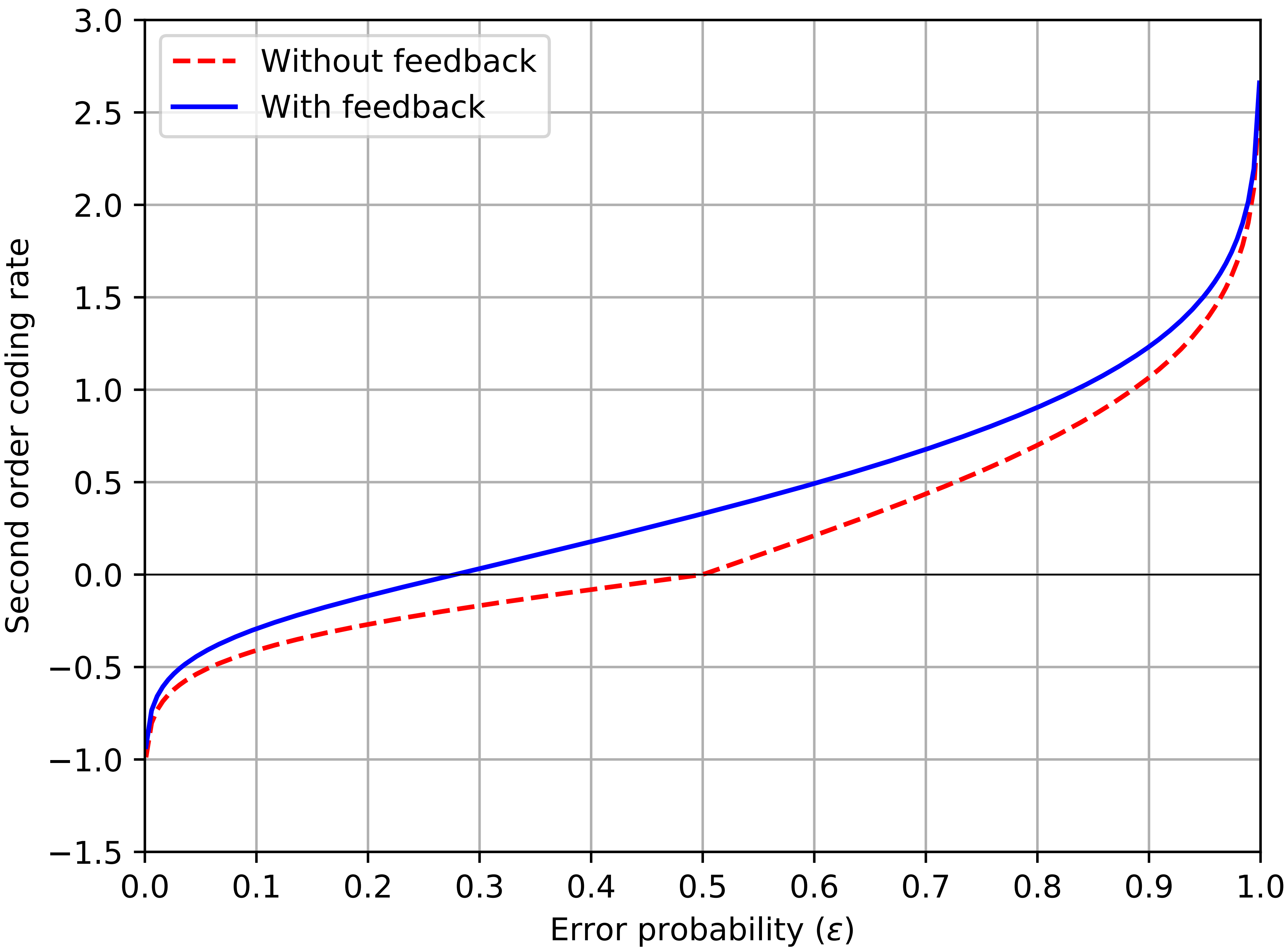
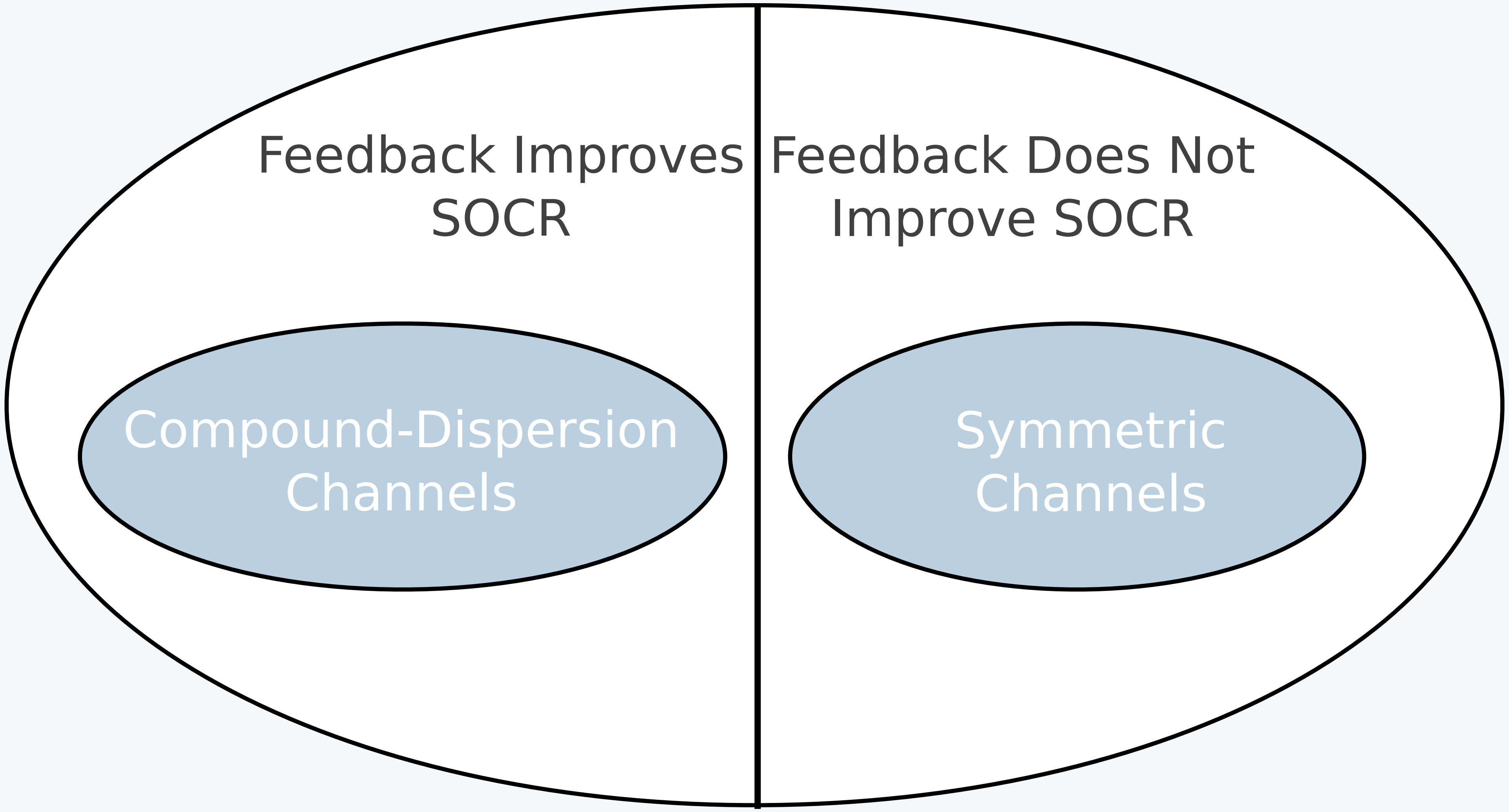
Numerical Example



When Does Feedback Help?



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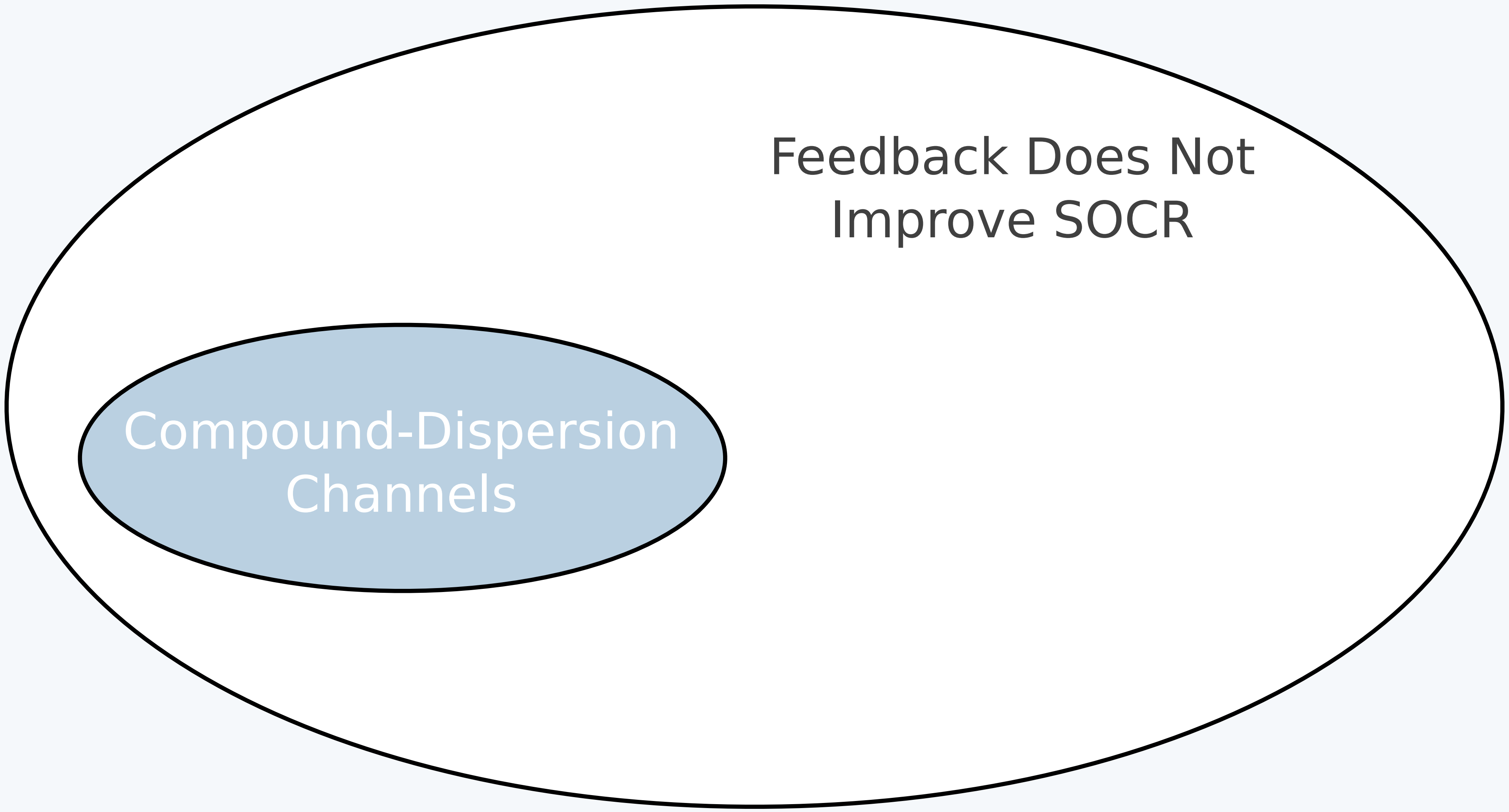

- **Theorem 2** (Wagner-Shende-Altug): Feedback improves the second-order coding rate iff the channel is compound dispersion.

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By How Much Does Feedback Help?

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[to apply McNamara,
need to switch to cont.-time]

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- **Theorem** (Strassen '67): If $\{S_n\}$ is a square-integrable martingale with $S_0 = 0$, then there exists a Brownian motion $B(\cdot)$ and a sequence of stopping times $0 = T_0 \leq T_1 \leq \ldots \leq T_n$ such that

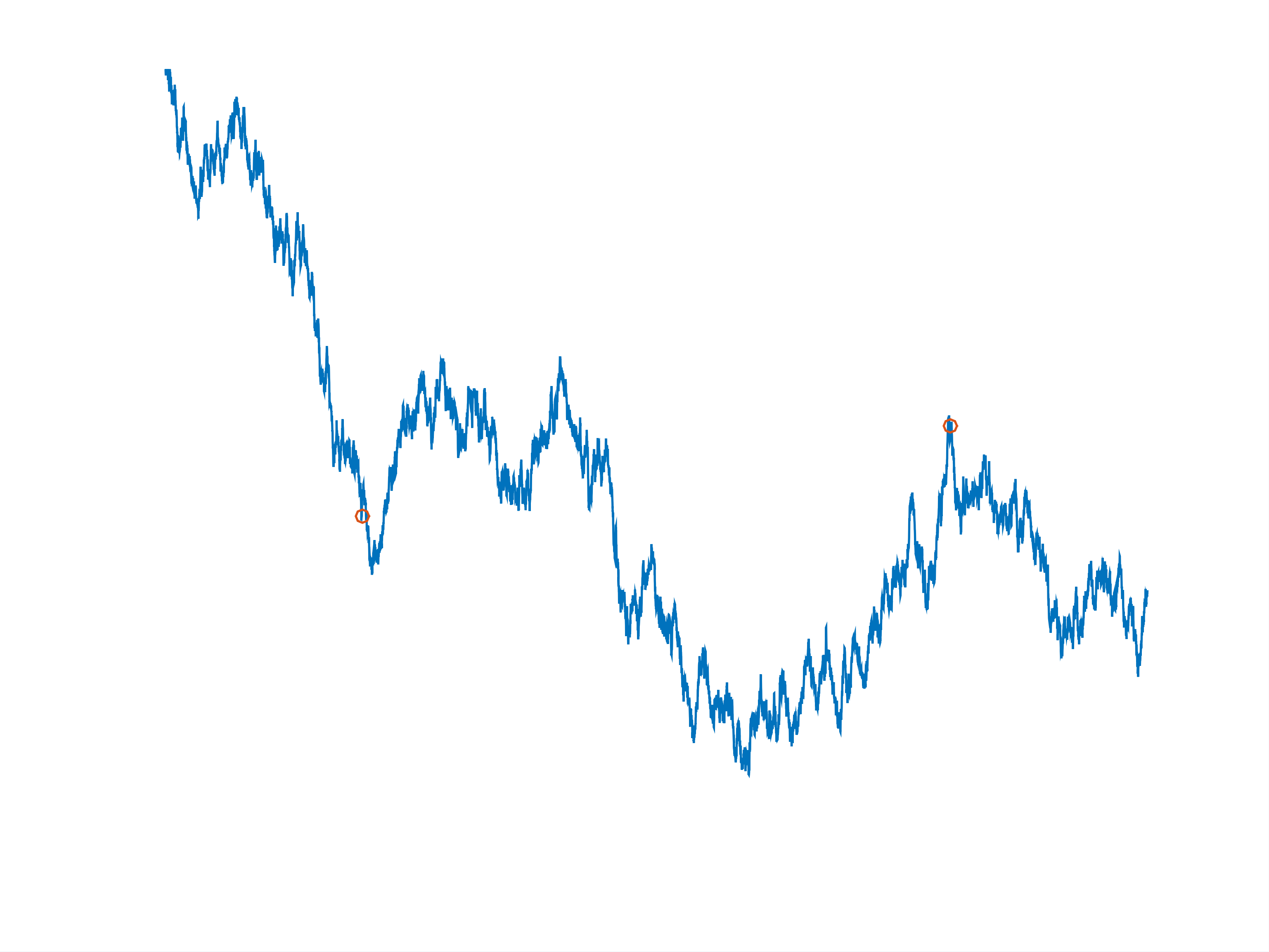
$$(S_0, S_1, \ldots, S_n) \stackrel{d}{=} (B(T_0), B(T_1), \ldots, B(T_n))$$

and

$$E[T_k - T_{k-1} | S_1, \ldots, S_{k-1}, T_1, \ldots, T_{k-1}]$$
$$= \mathrm{Var}(S_k - S_{k-1} | S_1, \ldots, S_{k-1})$$

Proof of Theorem 3

- So view *f* as selecting stopping times:



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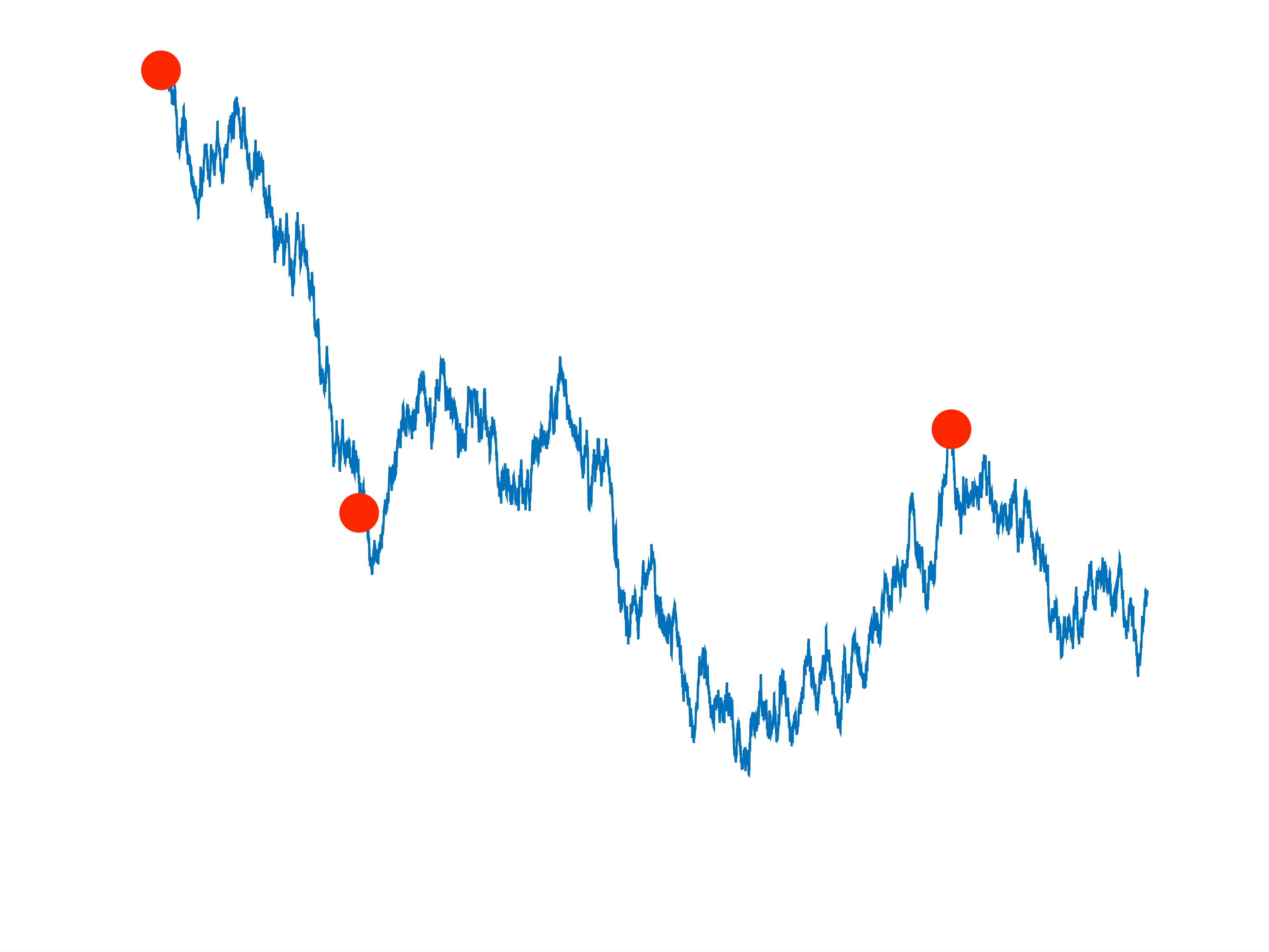

Proof of Theorem 3

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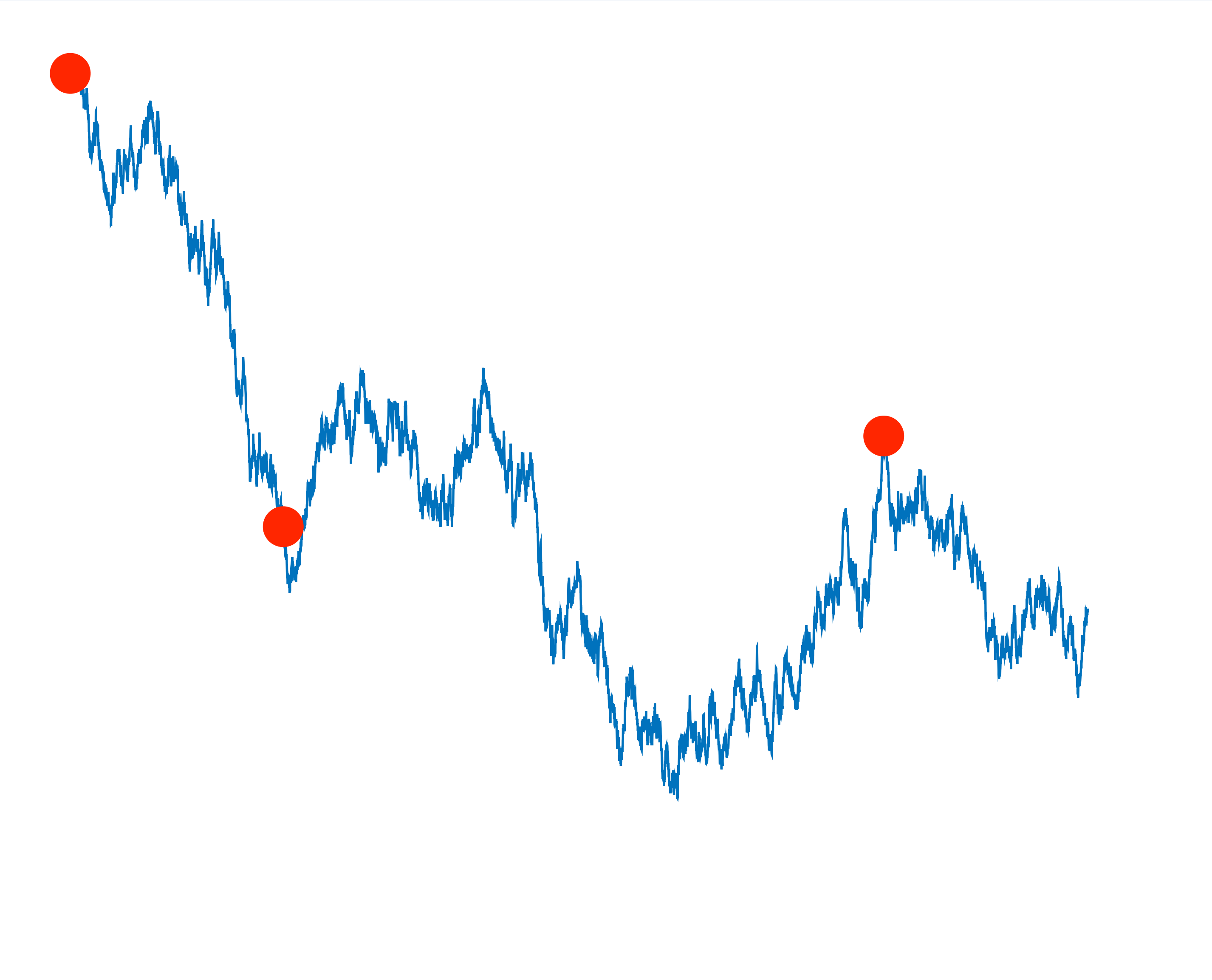

Proof of Theorem 3

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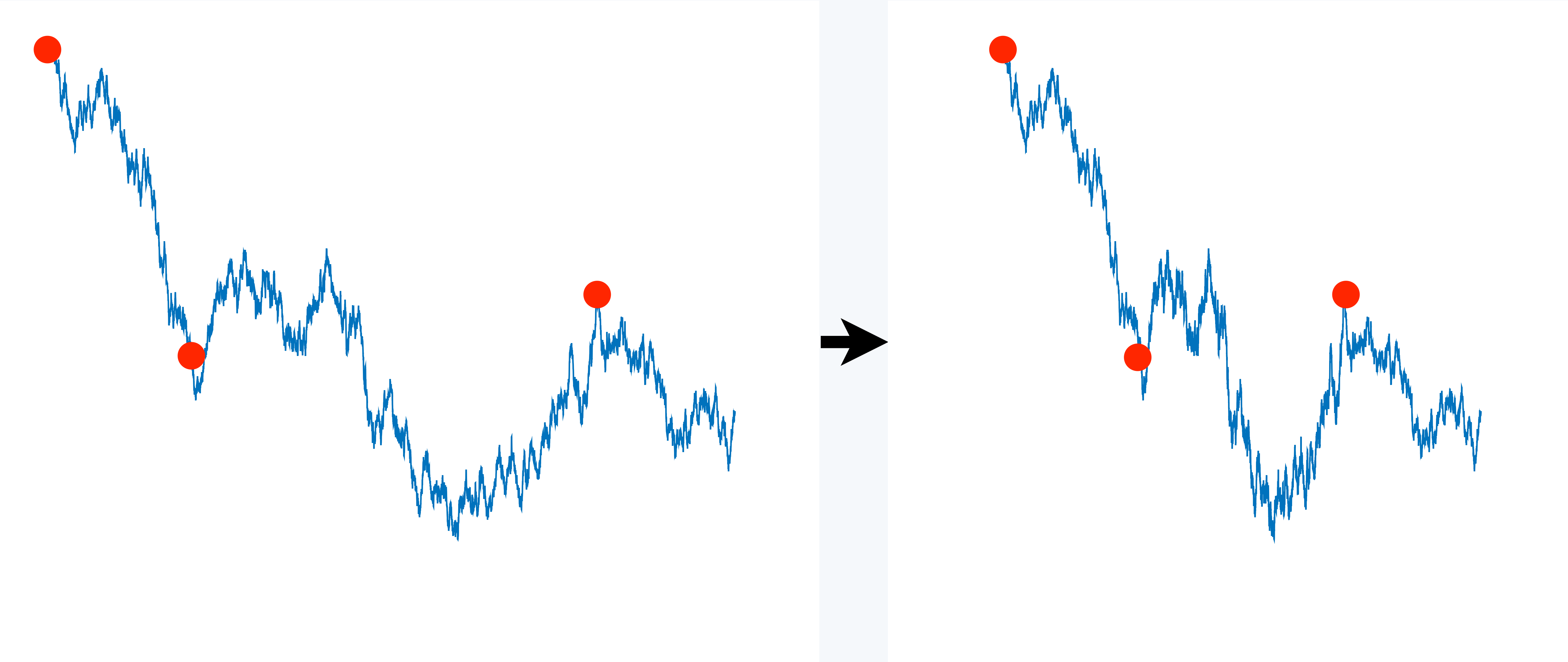

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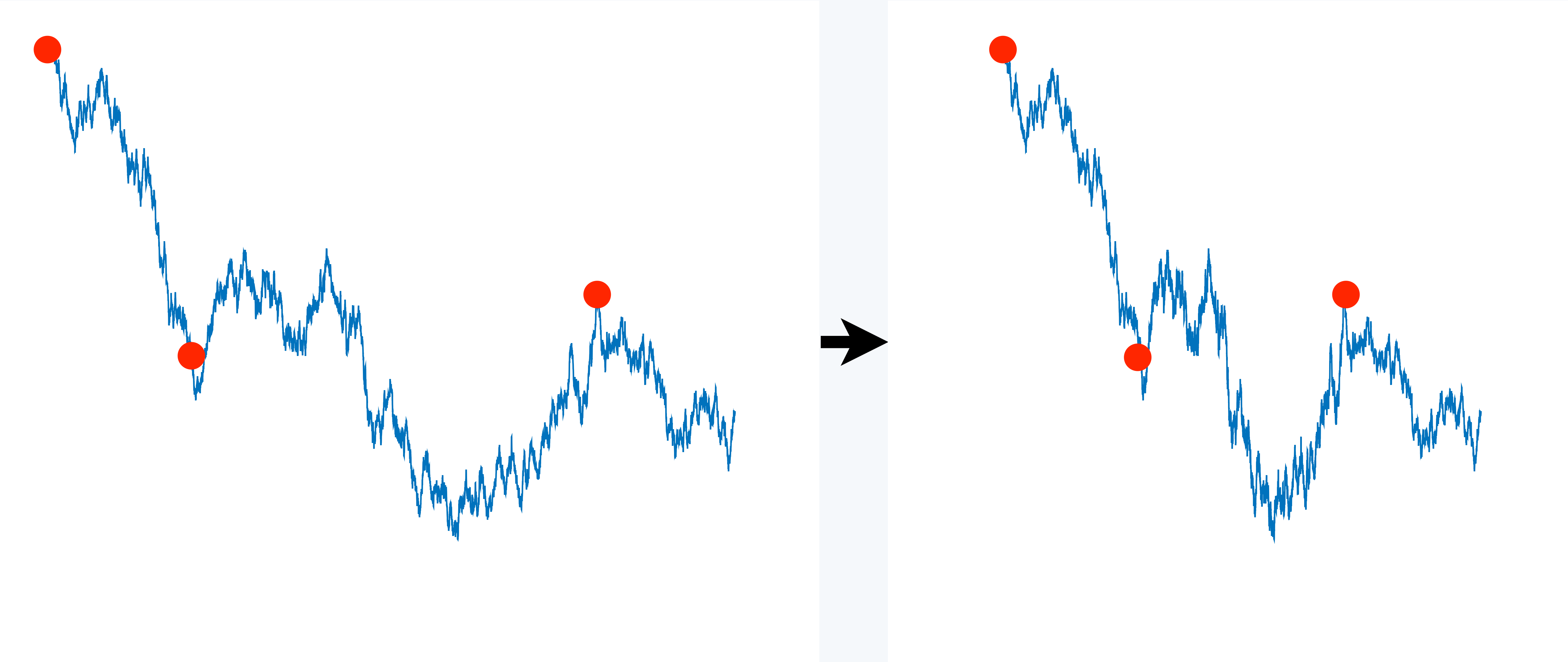

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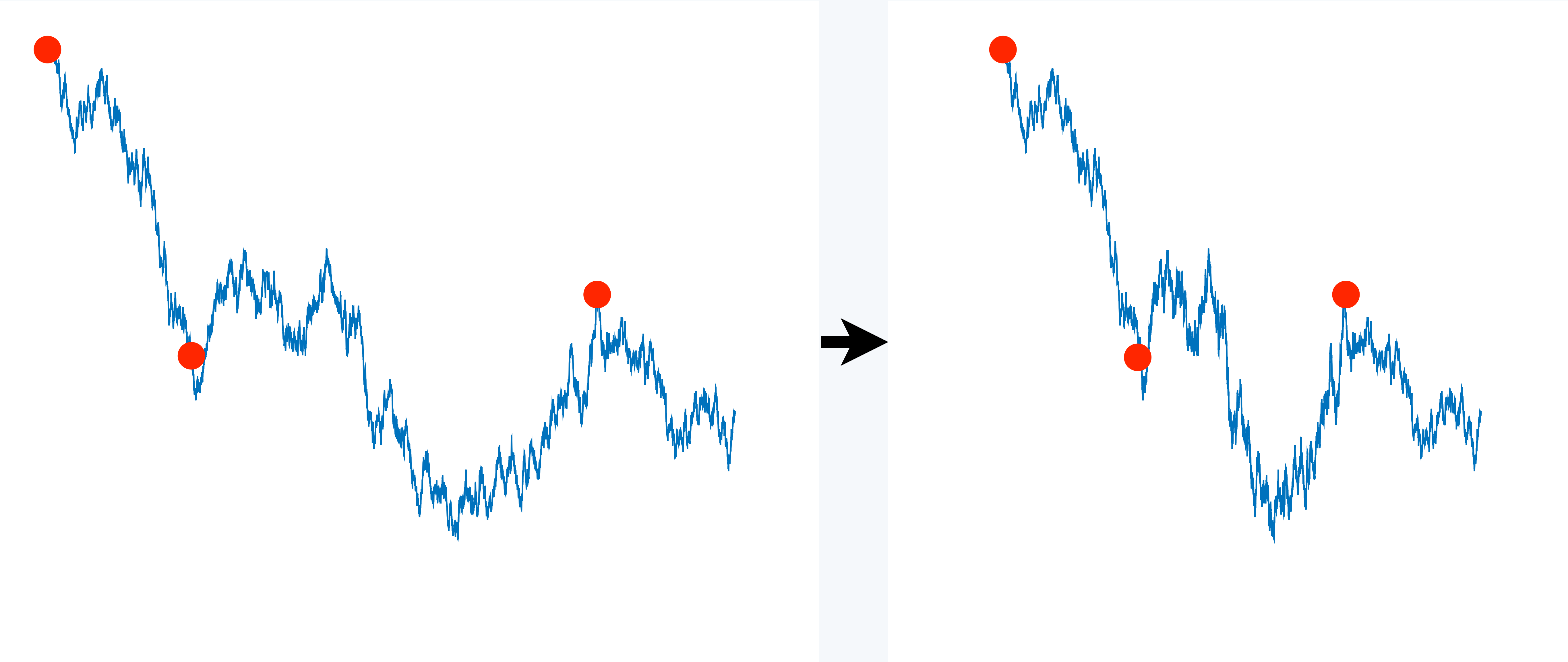

- Such *f* is *nearly* a feasible scheme in McNamara’s problem
- Make feasible and apply McNamara’s optimality result

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A Compound Dispersion Example

$$W(y|x) = \left[\begin{array}{ccc} p & 0.5(1-p) & 0.5(1-p) \\ 0.5(1-p) & p & 0.5(1-p) \\ 0.5(1-p) & 0.5(1-p) & p \\ \hline q & 1-q & 0 \\ 0 & q & 1-q \\ 1-q & 0 & q \end{array}\right]$$

$$\begin{array}{rl} \text{if} & p = 0.8 \\ \text{and} & q \approx 0.337 \\ \text{then} & V_{\min} = .102 \\ & V_{\max} = .692 \end{array}$$

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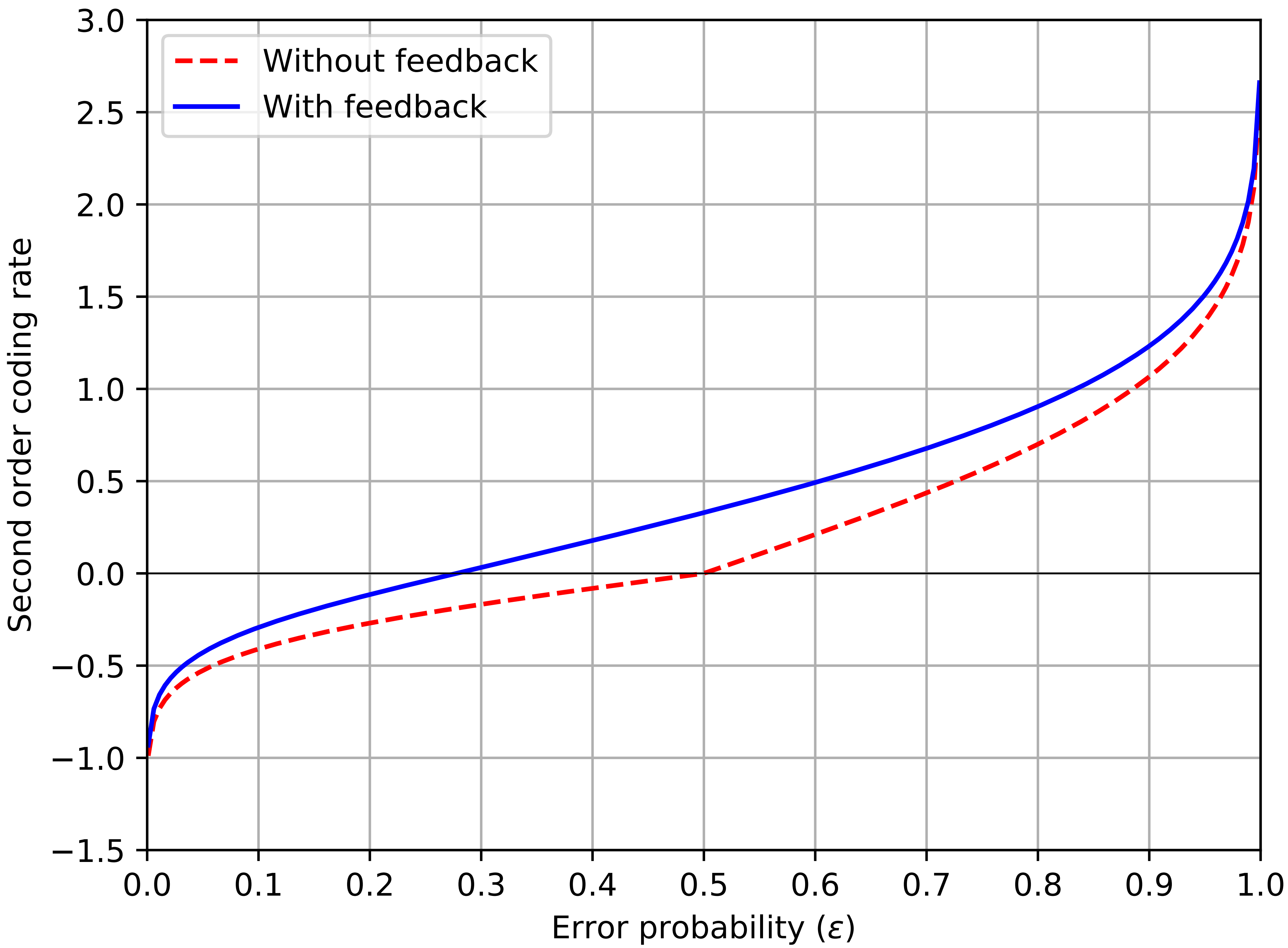
... so the upper bound is tight in this case.

Numerical Example



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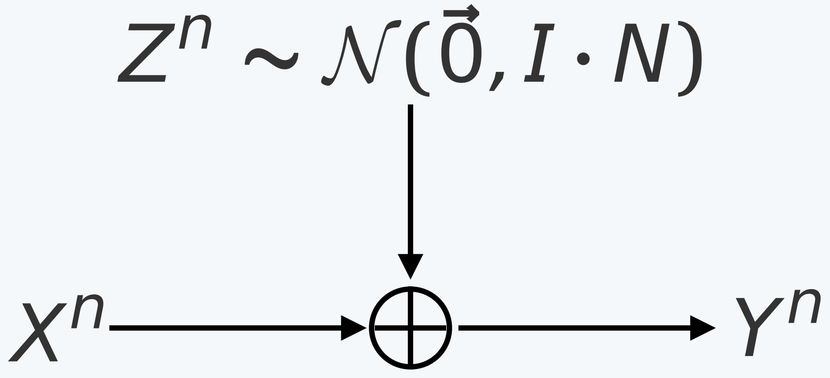

power constraint: P

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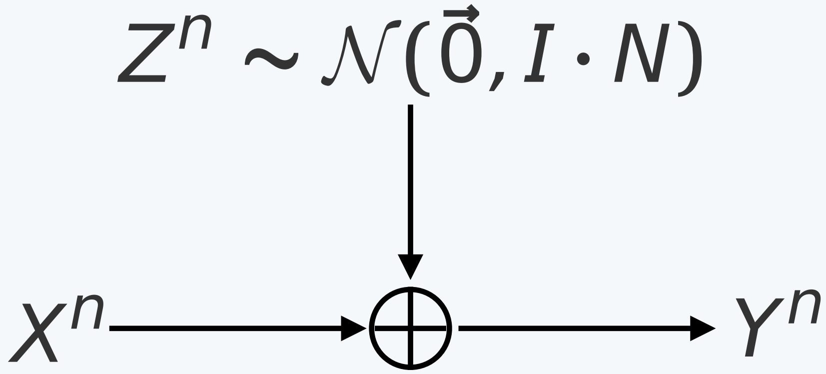

power constraint: P

If X^n is drawn uniformly from the radius-$\sqrt{nP}$ sphere:

$$\frac{1}{n}\mathrm{Var}\left[\log \frac{W(Y^n|X^n)}{Q^*(Y^n)}\right] = \frac{P(P+2N)}{2(P+N)^2}$$

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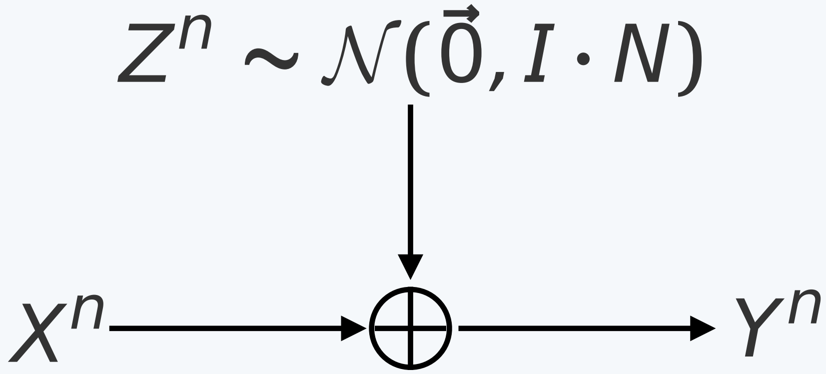

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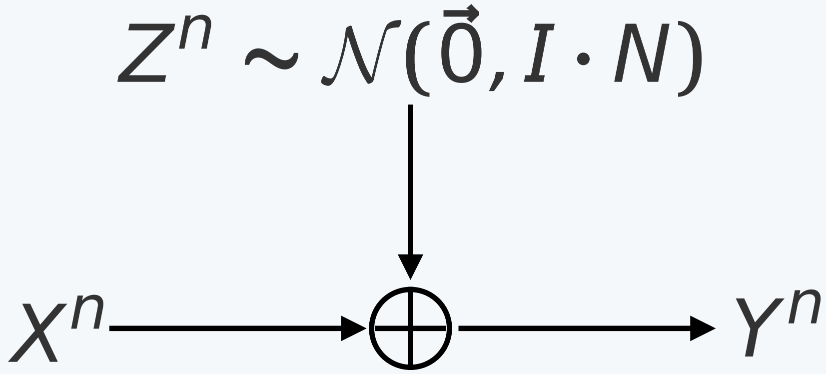

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[Similarly for any DMC with an active cost constraint]

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