How to distribute the multiplication of Secret Matrices?

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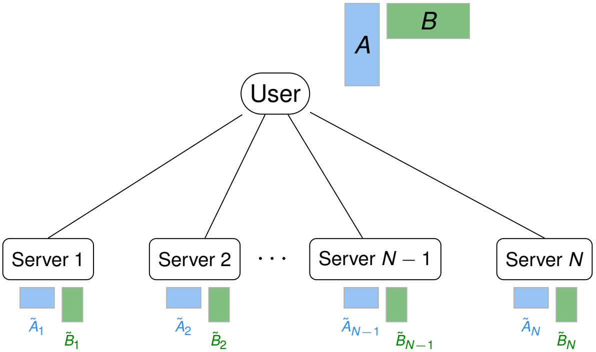
F-Secure

Setup



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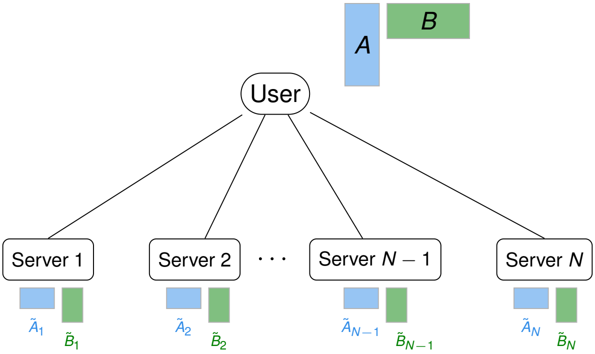

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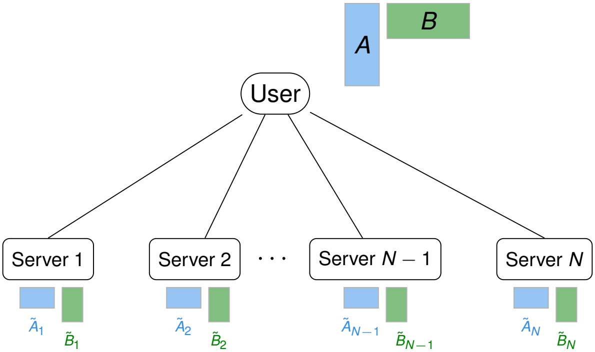

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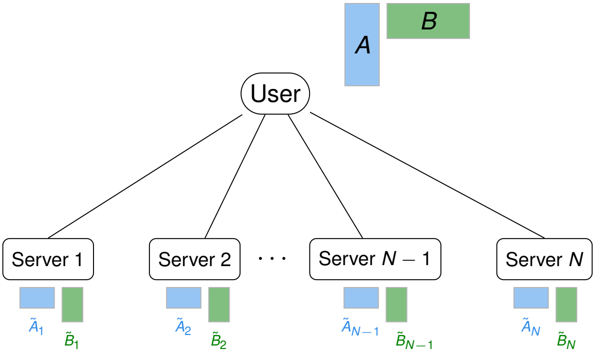

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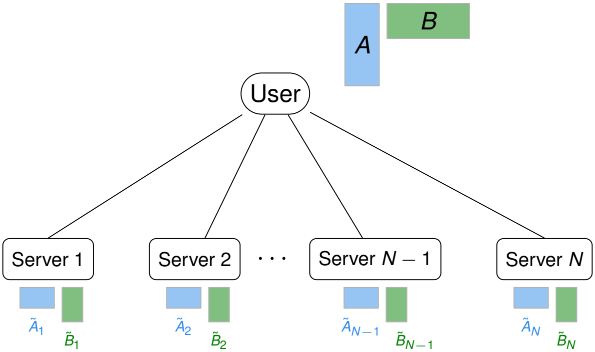

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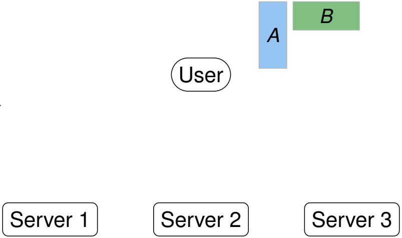
- Matrix Multiplication is everywhere!

Simplest Example: Polynomial Codes/Secret Sharing

$$f(x) = A + Rx$$
$$g(x) = B + Sx$$



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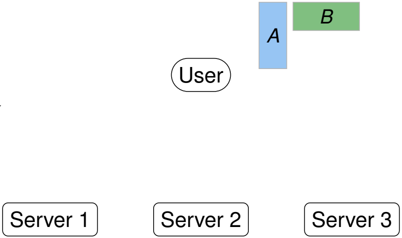

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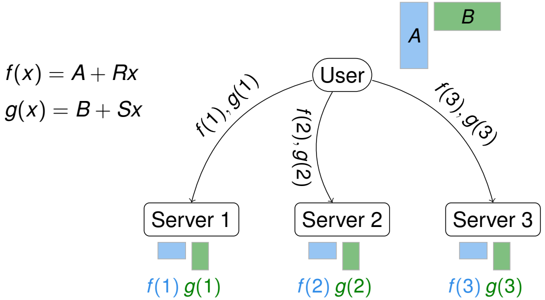

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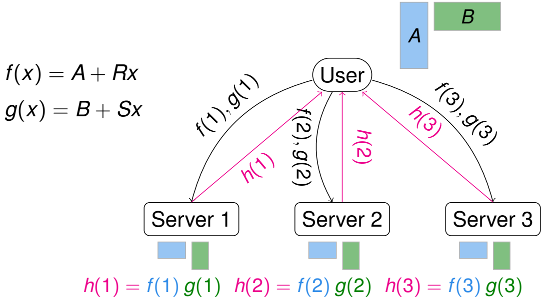

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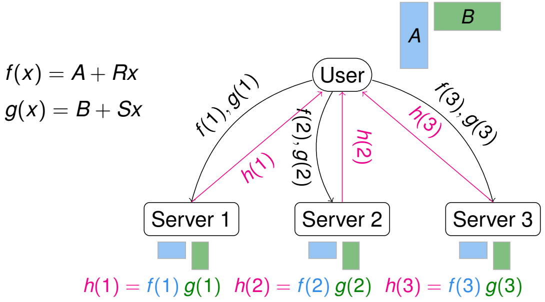

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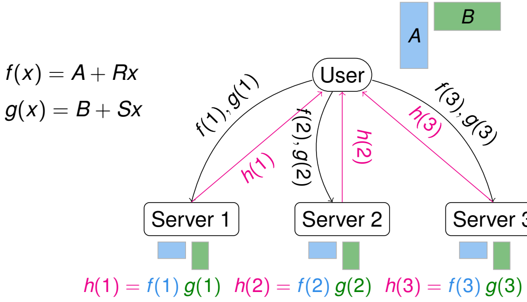

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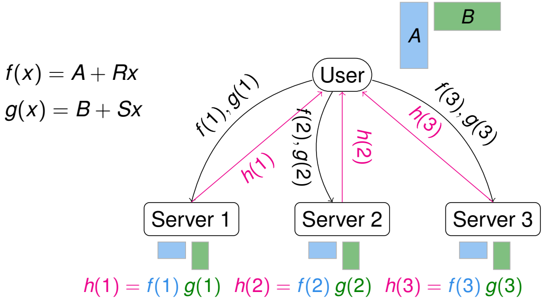

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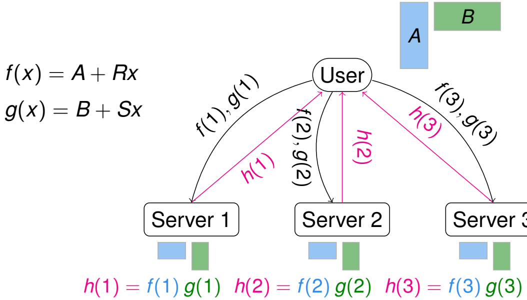

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- Server i computes $h(i) = f(i)g(i)$ and sends it to the user.
- User interpolates $h(x)$ and decodes $AB = h(0)$.
- Comm. cost $= 3 \times$ (upload A + upload B + download AB).

Divide & Parallelize

- Let $A \in \mathbb{F}_q^{r \times s}$ and $B \in \mathbb{F}_q^{s \times t}$

- We divide A and B as $A = \begin{bmatrix} A_1 \\ \vdots \\ A_K \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & \cdots & B_L \end{bmatrix}$.

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- $AB = \begin{bmatrix} A_1B_1 & \cdots & A_1B_L \\ \vdots & \ddots & \vdots \\ A_KB_1 & \cdots & A_KB_L \end{bmatrix}$
- Each server does $\frac{1}{KL}$ of the work.

Total Communication Cost

- When using N servers, the total Communication Cost is

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Goal: Given partition parameters K and L, and security parameter T, minimize the number of servers N.

Previous Work: Polynomial Codes for Stragglers

- Originally introduced in [Yu, Maddah-Ali, Avestimehr, '17].
- Different Setting: mitigating stragglers
- Other Work: [Yu, Maddah-Ali, Avestimehr, '18] ,
  [Dutta, Fahim, Haddadpour, Jeong, Cadambe, Grove, '18],
  [Sheth, Dutta, Chaudhari, Jeong, Yang, Kohonen, Roos, Grove, '18],
  [Li, Maddah-Ali, Yu, Avestimehr, '18],
  etc.

Previous Work: Polynomial Codes for Security

- Distributed multiplication with information theoretic security.
- [Chang, Tandon, ’18], [Kakar, Ebadifar, Sezgin, ’18] and [Yang, Lee, ’19]
- Related work: [Yu et al. ’19], [Aliasgari et al. ’19]

Can't Choose Any Polynomial

- Let $K = L = 3$ and $T = 2$.

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}, \quad AB = \begin{bmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{bmatrix}$$

- $f(x) = A_1 + A_2x + A_3x^2 + R_1x^3 + R_2x^4$
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$$h(x) = A_1B_1 + (A_1B_2 + A_2B_1)x + (A_1B_3 + A_2B_2 + A_3B_1)x^2 + \ldots$$

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- Can't retrieve A_1B_2, for example.

It is not about the degree.

- **Scheme 1:**
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  - $N_h = \deg h + 1 = 19$ servers.

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- $\deg h^* = 22 > 18 = \deg h$
- But h^* has gaps in the degrees.
- No term of degrees $13, 14, 16, 17$ or 20.
- Thus, only 18 points needed to interpolate h^*.
- $N_{h^*} = 18 < 19 = N_h$.

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- Not any points! What does $f(0)$ tell you?

How many terms does $f(x)g(x)$ have?

- $f(x) = A_1 x^{\alpha_1} + A_2 x^{\alpha_2} + A_3 x^{\alpha_3} + R_1 x^{\alpha_4} + R_2 x^{\alpha_5}$
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α_3	$\alpha_3 + \beta_1$	$\alpha_3 + \beta_2$	$\alpha_3 + \beta_3$	$\alpha_3 + \beta_4$	$\alpha_3 + \beta_5$
α_4	$\alpha_4 + \beta_1$	$\alpha_4 + \beta_2$	$\alpha_4 + \beta_3$	$\alpha_4 + \beta_4$	$\alpha_4 + \beta_5$
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α_4	$\alpha_4 + \beta_1$	$\alpha_4 + \beta_2$	$\alpha_4 + \beta_3$	$\alpha_4 + \beta_4$	$\alpha_4 + \beta_5$
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- We call this a degree table.

Properties of the Degree Table

- $f^*(x) = A_1 + A_2x + A_3x^2 + R_1x^9 + R_2x^{12}$
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h^*	0	3	6	9	10
0	0	3	6	9	10
1	1	4	7	10	11
2	2	5	8	11	12
9	9	12	15	18	19
12	12	15	18	21	22

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- **Goal:** Minimize distinct cells.

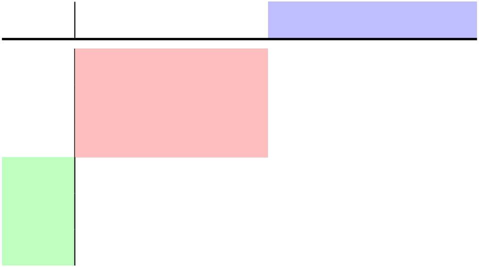
Problem Restatement: The Degree Table

	β_1	$\cdots$	β_L	β_{L+1}	$\cdots$	β_{L+T}
α_1	$\alpha_1+\beta_1$	$\cdots$	$\alpha_1+\beta_L$	$\alpha_1+\beta_{L+1}$	$\cdots$	$\alpha_1+\beta_{L+T}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_K	$\alpha_K+\beta_1$	$\cdots$	$\alpha_K+\beta_L$	$\alpha_K+\beta_{L+1}$	$\cdots$	$\alpha_K+\beta_{L+T}$
α_{K+1}	$\alpha_{K+1}+\beta_1$	$\cdots$	$\alpha_{K+1}+\beta_L$	$\alpha_{K+1}+\beta_{L+1}$	$\cdots$	$\alpha_{K+1}+\beta_{L+T}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_{K+T}	$\alpha_{K+T}+\beta_1$	$\cdots$	$\alpha_{K+T}+\beta_L$	$\alpha_{K+T}+\beta_{L+1}$	$\cdots$	$\alpha_{K+T}+\beta_{L+T}$

- **Goal:** Minimize number of distinct terms.
- Subject to:
  - Decodability: Numbers in the red region are all unique.
  - *A*-Security: Numbers in the green region are all distinct.
  - *B*-Security: Numbers in the blue region are all distinct.

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$



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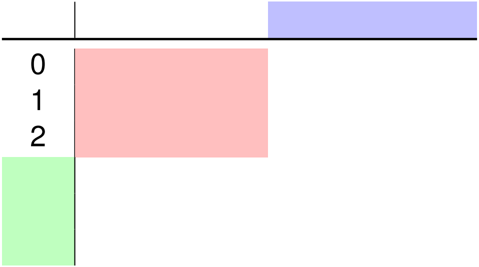

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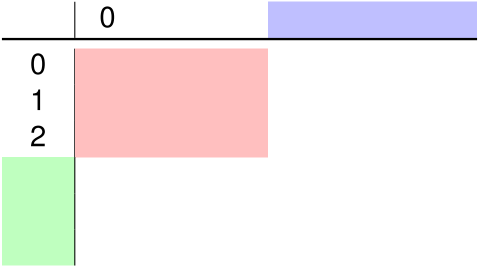

GASP_{big} [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0
0	0
1	
2	

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0
0	0
1	1
2	

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	
0	0	
1	1	
2	2	

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3
0	0	
1	1	
2	2	

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3		
0	0	3		
1	1	4		
2	2	5		

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	
0	0	3		
1	1	4		
2	2	5		

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6			
0	0	3	6			
1	1	4	7			
2	2	5	8			

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	
0	0	3	6	
1	1	4	7	
2	2	5	8	
9				

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	9		
0	0	3	6			
1	1	4	7			
2	2	5	8			
9						

GASP$_{big}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	9
0	0	3	6	9
1	1	4	7	10
2	2	5	8	11
9	9	12	15	18

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	9	10
0	0	3	6	9	
1	1	4	7	10	
2	2	5	8	11	
9	9	12	15	18	
10					

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$K = L = T = 3$

	0	3	6	9	10
0	0	3	6	9	10
1	1	4	7	10	11
2	2	5	8	11	12
9	9	12	15	18	19
10	10	13	16	19	20

GASP$_{\text{big}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	9	10	11
0	0	3	6	9	10	
1	1	4	7	10	11	
2	2	5	8	11	12	
9	9	12	15	18	19	
10	10	13	16	19	20	
11						

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$K = L = T = 3$

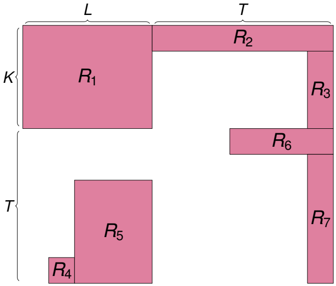
	0	3	6	9	10	11
0	0	3	6	9	10	11
1	1	4	7	10	11	12
2	2	5	8	11	12	13
9	9	12	15	18	19	20
10	10	13	16	19	20	21
11	11	14	17	20	21	22

Number of Terms

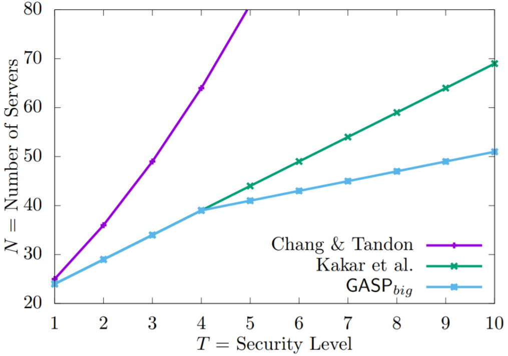
Theorem [D'Oliveira, SER, Karpuk, ISIT '19]

The number of terms in GASP_{big}, for $L \leq K$, is

$$N = \begin{cases} 2K + 2T - 1 & \text{if } L = 1 \\ (K+T)(L+1) - 1 & \text{if } L \geq 2,\, T < K \\ 2KL + 2T - 1 & \text{if } L \geq 2,\, T \geq K \end{cases}$$



How good is GASP$_{\text{big}}$?



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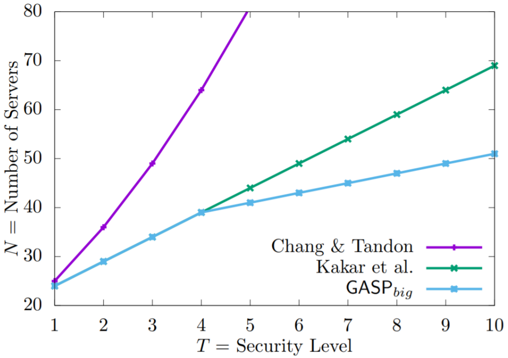

- Lagrange coding [Yu et al.,'19] achieves same rate for $T \geq \min\{K, L\}$.

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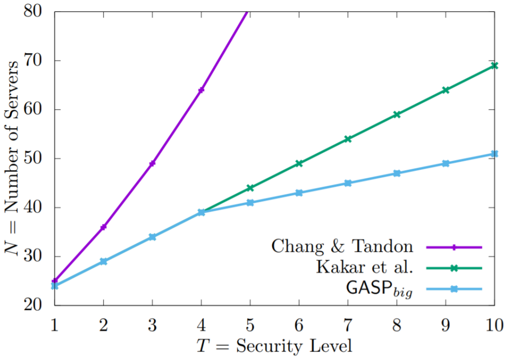

- Lagrange coding [Yu et al.,'19] achieves same rate for $T \geq \min\{K, L\}$.
- Can we do better?

GASP$_{\text{small}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	9	10	11
0	0	3	6	9	10	11
1	1	4	7	10	11	12
2	2	5	8	11	12	13

GASP$_{\text{small}}$ [D'Oliveira, SER, Karpuk, ISIT '19]

$K = L = T = 3$

	0	3	6	9	10	11
0	0	3	6	9	10	11
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9						

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	0	3	6	9	10	11
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1	1	4	7	10	11	12
2	2	5	8	11	12	13
9						
12						

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	0	3	6	9	10	11
0	0	3	6	9	10	11
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2	2	5	8	11	12	13
9						
12						
15						

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	0	3	6	9	10	11
0	0	3	6	9	10	11
1	1	4	7	10	11	12
2	2	5	8	11	12	13
9	9	12	15			
12	12	15	18			
15	15	18	21			

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	0	3	6	9	10	11
0	0	3	6	9	10	11
1	1	4	7	10	11	12
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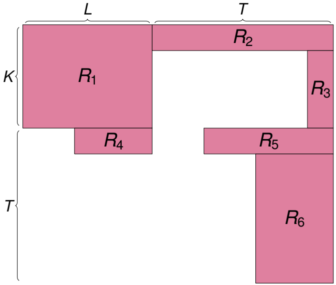
9	9	12	15	18	19	20
12	12	15	18	21	22	23
15	15	18	21	24	25	26

Number of Terms

Theorem [D'Oliveira, SER, Karpuk, ISIT '19]

The number of terms in $\text{GASP}_{\text{small}}$, for $K \leq L$, is

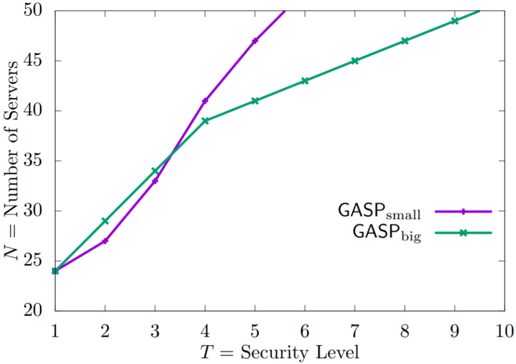
$$N = \begin{cases} 2K + T^2 & \text{if } L = 1, T < K \\ KT + K + T & \text{if } L = 1, T \geq K \\ KL + K + L & \text{if } L \geq 2, 1 = T < K \\ KL + K + L + T^2 + T - 3 & \text{if } L \geq 2, 2 \leq T < K \\ KL + KT + L + 2T - 3 - \left\lfloor \frac{T-2}{K} \right\rfloor & \text{if } L \geq 2, K \leq T \leq K(L-1) + 1 \\ 2KL + KT - K + T & \text{if } L \geq 2, K(L-1) + 1 \leq T \end{cases}$$



What is small T?

Theorem [D'Oliveira, SER, Karpuk, ISIT '19]

$\text{GASP}_{\text{small}}$ outperforms GASP_{big} for $T < \min\{K, L\}$.



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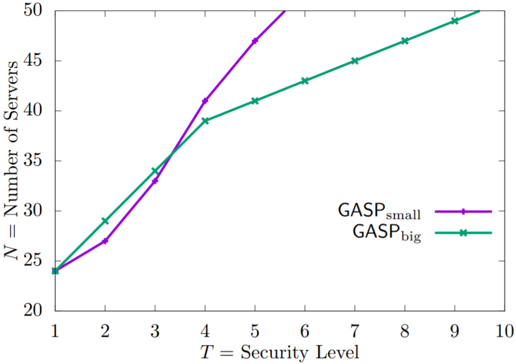

- Can we do better?

GASP$_r$: Gap Additive Secure Polynomial codes

	0	4	8	12	16	17	18	19
0	0	4	8	12	16	17	18	19
1	1	5	9	13	17	18	19	20
2	2	6	10	14	18	19	20	21
3	3	7	11	15	19	20	21	22
16	16	20	24	28	32	33	34	35
20	20	24	28	32	36	37	38	39
24	24	28	32	36	40	41	42	43
28	28	32	36	40	44	45	46	47

$r = 1,\ S(r) = 14,\ N = 41$

	0	4	8	12	16	17	18	19
0	0	4	8	12	16	17	18	19
1	1	5	9	13	17	18	19	20
2	2	6	10	14	18	19	20	21
3	3	7	11	15	19	20	21	22
16	16	20	24	28	32	33	34	35
17	17	21	25	29	33	34	35	36
20	20	24	28	32	36	37	38	39
21	21	25	29	33	37	38	39	40

$r = 2,\ S(r) = 19,\ N = 36$

	0	4	8	12	16	17	18	19
0	0	4	8	12	16	17	18	19
1	1	5	9	13	17	18	19	20
2	2	6	10	14	18	19	20	21
3	3	7	11	15	19	20	21	22
16	16	20	24	28	32	33	34	35
17	17	21	25	29	33	34	35	36
18	18	22	26	30	34	35	36	37
20	20	24	28	32	36	37	38	39

$r = 3,\ S(r) = 18,\ N = 37$

	0	4	8	12	16	17	18	19
0	0	4	8	12	16	17	18	19
1	1	5	9	13	17	18	19	20
2	2	6	10	14	18	19	20	21
3	3	7	11	15	19	20	21	22
16	16	20	24	28	32	33	34	35
17	17	21	25	29	33	34	35	36
18	18	22	26	30	34	35	36	37
19	19	23	27	31	35	36	37	38

$r = 4,\ S(r) = 16,\ N = 39$

Theorem [D'Oliveira, SER, Heinlein, Karpuk, ITW '19]

- Partitioning parameters: K and L
- Security parameter: T
- Chain length: r

Then, the degree table constructed by GASP_r has

$$N = KL + K + T - 1 + T \cdot (L + T) - S(r)$$

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where

$$S(r) = \max\{0, \min\{r, \varphi\}\}L + 2\max\{0, r - z + 1\} + \gamma + (T - r)L + \max\{0, K + T - KL - 1\} + \\ + \eta \max\{0, T - K + r - 1\} + (T - 1 - \eta)(T - 1)$$

$$\varphi = T - 1 - KL + 2K, \quad \eta = \lfloor (T-1)/r \rfloor, \quad z = \max\{1, \varphi + 1\},$$
$$\gamma = \begin{cases} 0 & \text{if } r < z \\ K(x - a)(x + a - 1)/2 - ab + xy + x & \text{else} \end{cases}$$

with a, b, x, y defined by

$$T - 1 - r = aK + b \text{ and } 0 \leq b \leq K - 1,$$
$$T - 1 - z = xK + y \text{ and } 0 \leq y \leq K - 1.$$

Lower Bounds

Theorem [D'Oliveira, SER, Heinlein, Karpuk, ITW '19]

- Partitioning parameters: K and L
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- Number of distinct terms: N

Then the following three inequalities hold.

1. $KL + \max\{K, L\} + 2T - 1 \leq N$.

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Then the following three inequalities hold.

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3. $KL + K + L + 2T - 1 - T\min\{K, L, T\} \leq N$.

Main Idea Behind Lower Bound

- Result from additive combinatorics on the minimal size of sum sets.

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Lemma [Tao,Vu, "Additive Combinatorics"]

Let A and B be sets of integers. Then $|A| + |B| - 1 \leq |A + B|$ and if $2 \leq |A|, |B|$, then equality holds iff A and B are arithmetic progressions with the same common difference.

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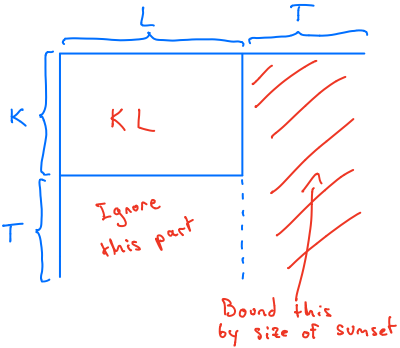
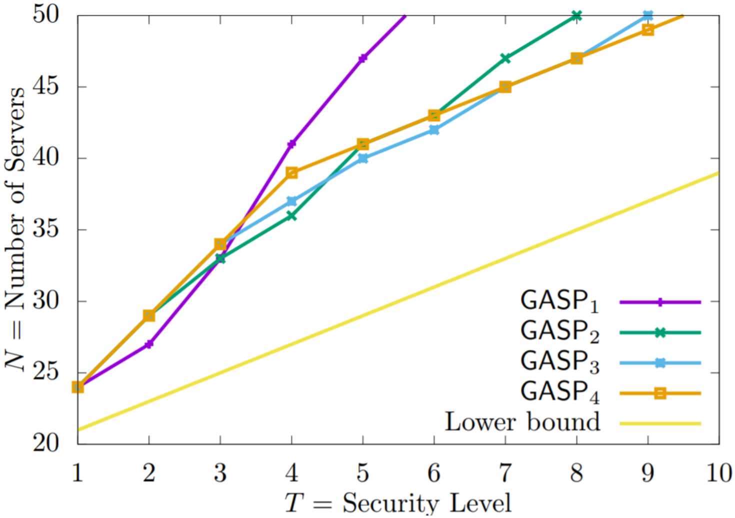


Current Situation

- $K = L = 4$
- GASP_r for $r = 1, \ldots, K$.



Optimality*

Corollary

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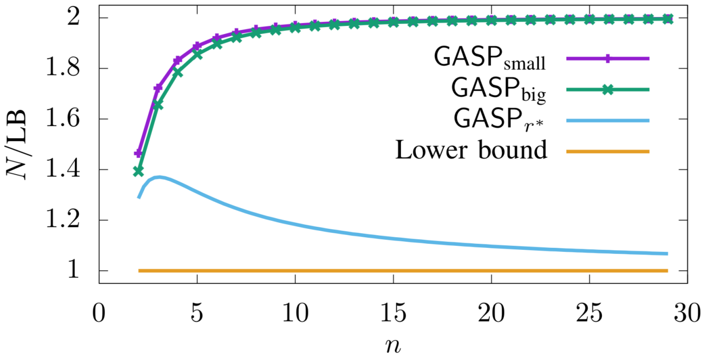

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Theorem [D'Oliveira, SER, Heinlein, Karpuk '20]

By using GASP, the user can perform the matrix multiplication in time $\mathcal{O}(n^{4-\frac{6}{\omega+1}}\log(n)^2)$ as opposed to the $\mathcal{O}(n^\omega)$ time it would take to do locally.

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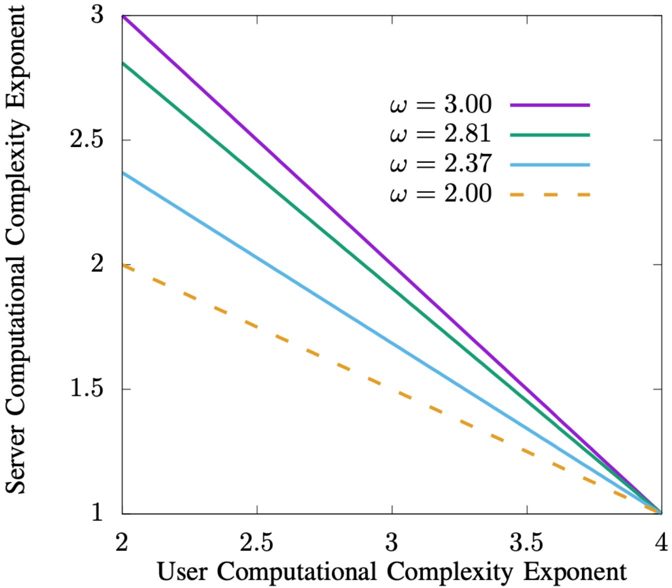

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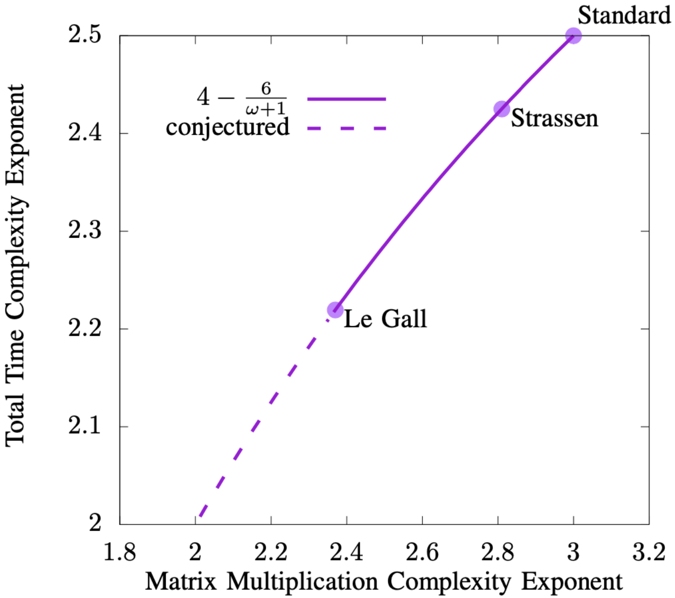

Open Problems

- Are there better schemes for the degree table?
- Are there better bounds?
- What about information theoretical bounds?
- Are polynomial codes optimal?

Thanks!