Network Coding in Minimal Multicast Networks

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***Abstract*— We investigate the network coding problem in a certain class of *minimal* multicast networks. In a multicast coding network, a source S needs to deliver h symbols, or packets, to a set of destinations T over an underlying communication network modeled by a graph G. A coding network is said to be h-minimal if it can deliver h symbols from S to the destination nodes, while any proper subnetwork of G can deliver at most $h-1$ symbols to the set of destination nodes. This problem is motivated by the requirement to minimize the amount of network resources allocated for a multicast connections.**

We show that, surprisingly, minimal multicast networks have unique properties that distinguish them from the general case of multicast networks. In particular, we show that it is possible to determine whether a 2-minimal network has a routing solution (i.e., a solution without encoding nodes) in polynomial time, while this problem is NP-hard in general. In addition, we show that if a 2-minimal network is planar, then the minimum size of the required field for linear network codes is at most 3. Also, we investigate several structural properties of 2-minimal networks and generalize our results for $h > 2$.

I. Introduction

A fundamental problem in the design of communication networks is to deliver information between the source and the destination nodes. Recently, it was shown that the information delivery can be facilitated by employing the novel technique of network coding [5]. The main idea of network coding is to allow intermediate nodes in the network to generate new packets by mixing the information received over their incoming links.

Multicast communication belongs to an important class of network communication problems. The goal of a multicast connection is to deliver h symbols from a source S to a set of t terminal nodes $T = \{T_1, \dots, T_t\}$.

This problem attracted a significant attention from researchers in the research community. In their seminal paper [5], Ahlswede et al. showed that network coding allows the network to achieve capacity in the case of a multicast communication. Li et al. showed [6] that the capacity can be achieved by using linear network codes, i.e., codes in which new messages outgoing from a node, are obtained by computing linear combinations, over a certain field, of the incoming symbols at the same node. Koetter and Médard [8] developed an algebraic framework for network coding. Ho et al. [7] showed that linear network codes can be efficiently constructed by employing a randomized algorithm. Jaggi et al. [11] proposed a deterministic polynomial-time algorithm for finding a feasible network code for a given multicast network. Rasala et al. proved [10] that the problem of finding the minimal size of a finite field over which a linear network code exists for a certain multicast problem is NP-hard.

In this work, we focus on the network coding problem for *minimal* multicast networks. A coding network (G, S, T) is said to be h-minimal if it can deliver h packets from S to T, while any proper subnetwork of G can deliver at most $h-1$ packets. The problem is motivated by the need of network service providers to minimize the amount of network resources allocated for individual multicast connections. Indeed, minimal multicast networks include only links that are essential for delivery of h packets to all T terminals, which minimizes the cost of establishing a multicast connection.

Contributions: The contribution of our paper can be summarized as follows. First, we analyze the complexity of deciding whether a given multicast problem admits a pure routing solution (i.e., a solution that does not require network coding). We show that this problem can be solved in linear time for 2-minimal coding networks. For the general case of non-minimal coding networks, this problem was shown to be NP-hard [12]. We present here another proof of this result based on a reduction from the problem of vertex coloring of multigraphs.

We also show that all network coding problems in 2-minimal networks have a similar structure. Specifically, any such problem can be reduced to the problem of finding a network code for a two-layer network, with a single coding node located at the source. Moreover, we describe a family of 2-minimal networks that admit a polynomial time algorithms for finding the minimal field size over which a linear network code exists. Next, we consider planar networks, which are often encountered in many practical settings. We show that network codes for 2-minimal planar networks can be found over any field of size 3. Finally, we consider the case of h-minimal networks for $h > 2$ and present a certificate that allows to verify, in an efficient way, that a given network does not admit a pure routing solution, in the special case when the network does not contain nodes of in-degree 2.

The rest of the paper is organized as follows. In Section II, we formally define our model. In Section III, we analyze the structure of 2-minimal coding networks. In Section IV, we present a family of 2-minimal coding networks in which the minimum size of the field can be computed by a polynomial time algorithm, and discuss some properties of planar networks. In Section V, we present the algorithm that

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determines whether a 2-minimal coding network has a pure routing solution and show that this problem is NP-hard in general. In addition, we analyze the properties of h-minimal coding networks for $h > 2$. Finally, we summarize our work and draw some conclusions in Section VI.

II. MODEL

The underlying communication network is represented by an acyclic directed graph $G = (V, E)$ where V is the set of nodes and E the set of links. We assume that each link $e \in E$ can transmit one symbol per time unit. In order to model links whose capacity is higher than one unit, G may include multiple parallel links. A coding network problem $\mathbb{N}(G, S, T)$ is a 3-tuple that includes the graph G, a source node $S \in V$, a set of terminals or sinks T. We assume that each packet is a symbol of some alphabet Σ.

The capacity of a multicast coding network is defined to be the maximum number of packets that can be delivered from S to T, and is determined by the minimum value of a cut that separates the source S and any terminal $T_i \in T$ [6]. A network code for $\mathbb{N}$ is said to be *feasible* if it *allows communication at rate* h between S and each terminal $T_i \in T$, where h is the capacity of the network. The existence of a feasible network code was shown in [6].

We proceed by introducing the notion of minimal communication networks [12], [10], [9].

Definition 1 (Minimal Coding Network): A coding network $\mathbb{N}(G, S, T)$ is said to be h-minimal if its capacity is h, and if the capacity of any network $\hat{\mathbb{N}}(\hat{G}, S, T)$ formed from G by deleting a link e from G is at most $h - 1$.

We first observe that in an h-minimal network, the degree of any node in the network is at most h. We refer to all nodes of in-degree 1 as *Steiner* nodes, and all other nodes in the network, except the source, as *non-Steiner* nodes. We assume, without loss of generality, that in h-minimal networks there are no adjacent Steiner nodes. Indeed, any two such nodes can be combined together, by contracting the edge that connects them, resulting in an equivalent network.

III. STRUCTURE OF 2-MINIMAL NETWORKS

In this section we show that all network coding problems in 2-minimal networks have a similar structure. To that end, we show a reduction from a problem of finding a feasible network code for a 2-minimal network to the problem of finding a feasible network code for a network that belongs to the family of $M_{s,t}$ networks, defined below.

An $M_{s,t}$ network is a two-layer bipartite network that includes s nodes $S_1, S_2, \ldots, S_s$ that belong to the first layer and t sink nodes $T_1, T_2, \ldots, T_t$ that belong to the second layer. The network also includes a special source node S that is connected to each intermediate node $S_1, S_2, \ldots, S_s$. In addition, each sink node is connected to two different intermediate nodes S_i and S_j. Figure 1 demonstrates an example of an $M_{s,t}$ network.

The following theorem shows a correspondence between 2-minimal coding networks and $M_{s,t}$ networks.

Theorem 2: Let $\mathbb{N}(G, S, T)$ be a 2-minimal coding network. Then, there exists an $M_{s,t}$ network $\mathbb{N}'(G', S', T')$ that has the following properties:

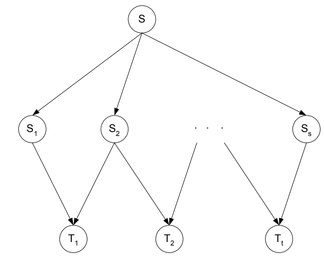

Fig. 1. The graph G' corresponding to the $M_{s,t}$ family of multicasting problems.

1) $\mathbb{N}'(G', S', T')$ is a 2-minimal network;
2) The parameter s is equal to the number of Steiner nodes of G, while t is less or equal to the number of non-Steiner nodes of G.
3) For any feasible linear network code C' for $\mathbb{N}'$, there exists a feasible linear network code C for $\mathbb{N}(G, S, T)$, over the same field, that can be found in polynomial time; and *vice versa*.

Proof: Given $\mathbb{N}(G, S, T)$, we construct $\mathbb{N}'(G', S', T')$ as follows. First we index all nodes of G in topological order, i.e., for every link (v, u) the label of v is smaller than that of u. Such an ordering is possible due to the fact that the network G is acyclic. Next, we process each node v of G in topological order. If node v is a Steiner node S_i, we remove the link between v and its predecessor and connect it directly to the source S. If v is a non-Steiner node such that at least one of its predecessors is also a non-Steiner node, then we remove v and all links incident to it from the network. For all other nodes, no changes are made. Finally, we define $S' = S$ and T' to be the set of all non-Steiner nodes in G'. C' can be obtained in the following manner: each Steiner node in G' requests from the source the same message that flows through its corresponding Steiner node in G when the network code C is applied.

Next, we show that for any feasible network code C' for $\mathbb{N}'$, we can construct efficiently a feasible network code C for $\mathbb{N}(G, S, T)$. Let C' be a feasible code of $\mathbb{N}'$ and let m_1 and m_2 be the two symbols to be conveyed to the sinks. We show, by induction on the topological order of the vertices of G, that any symbol received by a node $v' \in G'$ can be received by the corresponding node $v \in G$, and that all the non-Steiner nodes of G that were deleted when constructing $\mathbb{N}'$ can reconstruct both symbols m_1 and m_2. Clearly, this holds for nodes directly connected to the source S. Now assume this is true for all nodes in G of index less or equal than $n-1$, and consider the node $v \in G$ of index n. If v is a Steiner node, it requests the same message as the one that its corresponding node in $v' \in G'$ is getting from S'. The predecessor of v can satisfy his request because it is either the source S, or a non-Steiner node that, by the induction assumption can decode the two packets and thus form any linear combination of them.

If v is a non-Steiner node, then we consider three cases. If both of v's predecessors are Steiner nodes, then by construction, it can receive the same messages as the corresponding node v' in G'. Suppose that v is connected to two non-Steiner nodes. By the induction hypothesis, both of these nodes can reconstruct both m_1 and m_2 and send them to v. Finally, suppose that one of v's predecessors, is a Steiner node and second is a non-Steiner node. The non-Steiner node can reconstruct both m_1 and m_2 or any linear combination of them. Thus, this node can generate a message which is linearly independent of the message sent by the Steiner node predecessor of v. We conclude that v can reconstruct both packets m_1 and m_2, which completes the proof of the theorem. ■

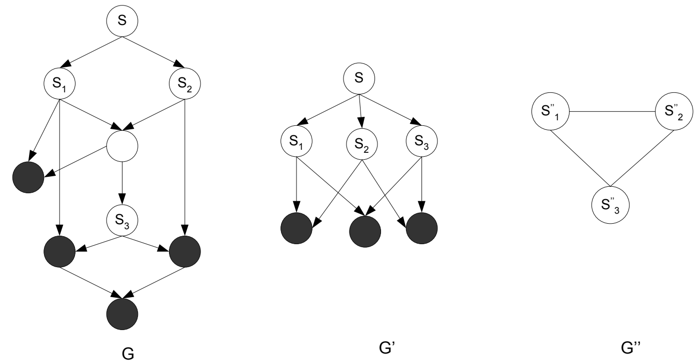

Fig. 2. Construction of the graphs G' and G'' from the graph G. Sinks are represented by black nodes.

IV. Field Size

Let q_{min} be the minimum size of a finite field required by a feasible network code for a given 2-minimal coding network $\mathbb{N}(G, S, T)$. It was shown by Rasala et al. [10] that computing q_{min} is an NP-hard problem. In this section we present a family of 2-minimal coding networks in which the value of q_{min} can be computed in polynomial time. We also show that if $h = 2$ and G is planar, than q_{min} is upper bounded by 3.

We begin by defining an auxiliary graph G''. The graph G'' is similar to that used by the reduction described in [10], and is constructed as follows. First, for each Steiner node $v \in G$, we add a corresponding node v'' to G''. Then, for any two Steiner nodes $v, u \in G$ that have a common child node in G, we connect the two corresponding nodes v'' and u'' by an edge in G''. It was shown in [10] that $q_{min} = \chi(G'') - 1$, where $\chi(G'')$ is the chromatic number of G''.

Definition 3: We say that a 2-minimal coding network $\mathbb{N}(G, S, T)$ is *transitive* if it satisfies the following condition: For any three Steiner nodes v, u and w in $\mathbb{N}$, it holds that if v and u have a common child and u and w have a common child then v and w have a common child.

Theorem 4: For any transitive network $\mathbb{N}(G, S, T)$, the value of q_{min} can be computed in $O(|E| + |V|)$ time.

Proof: We begin by constructing the auxiliary graph G'' corresponding to G, as defined above. G'' will have the following property: for any three nodes v'', u'', and w'', if there is an edge between v'' and u'', and an edge between u'' and w'', then there is an edge between v'' and w''. Hence each cycle of G'' of length at least four has a chord. Therefore, G'' is a triangulated graph [2, Chapter 4]. By Theorem 4.17 of [2], we know that the chromatic number of G'' can be calculated in $O(|V''| + |E''|)$ time. Thus, since $|E| = O(|E''|)$ and $|V| = O(|V''|)$, q_{min} can be computed in $O(|E| + |V|)$ time. ■

Theorem 5: If G is planar then $q_{min} \leq 3$.

Proof: It can be easily seen that G'' can be obtained from G in following three steps:

1) Delete from G the source node and all sinks that have at least one non-Steiner node parent and all their adjacent edges.
2) Delete all edges that connect Steiner nodes to their parents.
3) For each remaining non-Steiner node v, contract an edge that connects v to one of its parents.

It is clear that if we start from a planar graph, then the one resulting after the first two steps is planar too. Also, contracting edges in a planar graph would result in a planar graph since planarity is a minor closed property. Therefore, the graph G'' obtained after the third step is also planar.

Then, by the famous Four Colors Theorem [1, Theorem 5.5.1], G'' can be colored using only four colors. Therefore $\chi(G'') \leq 4$ and $q_{min} = \chi(G'') - 1 \leq 3$. ■

Note that if G is not planar but has a planar 2-minimal subgraph, then it is also the case that $q_{min} \leq 3$.

V. Routing Solutions

At the end of their seminal paper [5], the authors ask the following question:

> "Also, we can ask under what condition can optimality be achieved without network coding."

We say that a coding network has a routing solution if its capacity can be achieved without network coding. In this section, we address this question by investigating the conditions under which a routing solution exists for a given coding network $\mathbb{N}(G, S, T)$. The problem of finding the maximal throughput that can be achieved without network coding in a general multicast network is equivalent to as the Steiner packing problem, and is known to be an NP-hard problem ([13], [14]).

We denote by $d^-(v)$ and $d^+(v)$ the in- and out- degrees of vertex v, respectively. We observe that if there is a routing solution for $\mathbb{N}(G, S, T)$, where $\mathbb{N}(G, S, T)$ is h-minimal, then $d^-(v) \leq d^+(v)$ for all $v \in G \setminus T$. We also observe that the converse is not always true. For example, consider the problem corresponding to graph G', in Figure 2, whose all nodes, except the sinks, have in-degrees less then out-degrees, but it does not have any routing solution.

Lemma 6: Let $\mathbb{N}(G, S, T)$ be a 2-minimal coding network and $\mathbb{N}'$ be the corresponding $M_{s,t}$ network, as defined in the proof of Theorem 2. Then $\mathbb{N}$ has a routing solution if and only if one exists for $\mathbb{N}'$.

Proof: Follows immediately from our construction of a code for $\mathbb{N}$ given one for $\mathbb{N}'$, and vice versa, in the proof of Theorem 2. ∎

Thus, we can assume here that $\mathbb{N}(G, S, T) \in M_{s,t}$ without any loss of generality.

Theorem 7: Let $\mathbb{N}(G, S, T)$ be a 2-minimal coding network. Then, there exists a routing solution for $\mathbb{N}(G, S, T)$ if and only if G'' is a bipartite graph.

Proof: Assume that we have a routing solution for $\mathbb{N}(G, S, T) \in M_{s,t}$. Then the message that flows through the intermediate nodes of G is one of the two symbols to be transmitted, either m_1 or m_2. We associate with each symbol m_i a different color c_i. And we color G'' in the following way. For each intermediate node S_i $(i = 1, 2, \ldots, s)$,if symbol m_j $(j = 1, 2)$ flows through it, we color the corresponding node S''_i in G'' with the color c_j. Two adjacent nodes in G'' are not colored by the same color, otherwise this would imply that a sink in G is receiving the same symbol on both incoming edges. Thus, this coloring of G'' is indeed a proper coloring. Since G'' is two colorable, than it is bipartite.

The converse can be similarly proven by associating with each one of the two colors of G'' a different symbol. Then a routing solution of $\mathbb{N}(G, S, T)$ is constructed by letting the symbol that flows through an intermediate node in G be the one associated with the color of the corresponding node in G''. ∎

Corollary 8: Deciding whether $\mathbb{N}(G, S, T)$ has a routing solution, and if so finding this solution, can be done in $O(|E| + |V|)$ time.

Proof: By Theorem 7, we can check whether $\mathbb{N}(G, S, T)$ has a routing solution by checking if G'' is bipartite. Since, bipartite graphs are characterized by the property that they do not have any odd cycle [1, Proposition 1.6.1], then one can use a slightly modified version of the BFS (Breadth-First-Search) algorithm [4, Section 22.2] to decide whether G'' is bipartite and if so obtain a partition of V into two *independent* sets [15]. The result directly follows from the fact that G'' can be constructed from G in $O(|E| + |V|)$, and that the BFS algorithm runs in $O(|E| + |V|)$ time. ∎

In the remainder of this section, we investigate whether the above results still hold for more general cases. In particular, the following theorem shows that even if we restrict ourselves to problems $\mathbb{N}(G, S, T)$ of multicasting two symbols over a network G, but do not require the G to be minimal, then deciding if $\mathbb{N}(G, S, T)$ has a routing solution is an NP-hard problem.

Theorem 9: It is NP-hard to decide whether a general two symbol multicast network coding problem has a routing solution.

Proof: We use a reduction from vertex coloring of hypergraphs.

Let $\mathbb{H}(\mathbb{V}, \mathbb{E})$ be a hypergraph (edges here are subsets of $\mathbb{V}$ [3]). We can assume, without loss of generality, that $\mathbb{H}$ contains no loops (edges of cardinality one), since they do not play any role in coloring problems. We map $\mathbb{H}$ into the network problem $\mathbb{N}(G, S, T)$ of multicasting two symbols over a graph $G(V, E)$. Graph G consists of a source node S and an intermediate node S_i for each vertex $V_i \in \mathbb{V}$. For each different edge $E_j \in \mathbb{E}$ we add to G a sink T_{E_j} connected by incoming edges to all nodes $V_i \in E_j$. Figure 3 depicts a hypergraph $\mathbb{H}$ and the corresponding graph G resulting from this reduction. Note that G is not necessarily minimal since $\mathbb{H}$ might contain hyperedges connecting more that two vertices.

We show that $\mathbb{H}$ is 2-colorable if and only if $\mathbb{N}(G, S, T)$ has a routing solution. First, we color the vertices of $\mathbb{H}$ using only 2 colors. We map each of the two colors c_i bijectively to one of the two symbols m_i. A routing solution of $\mathbb{N}(G, S, T)$ is then obtained by letting the intermediate node S_i send to all its children the symbols m_i, where c_i is the color of $V_i \in \mathbb{H}$. This way, all sinks receive both symbols, since there is no monochromatic edge in $\mathbb{E}$.

Conversely, suppose that $\mathbb{N}(G, S, T)$ has a routing solution. We map each of the symbols m_i bijectively to two different colors c_i. then, we color each $V_i \in \mathbb{V}$ by c_i if the message that is flowing through S_i is m_i. Then, for all vertices in $\mathbb{H}$ that are left uncolored, we color them arbitrarily. Since each sink will be receiving two different packets, then we have no monochromatic edges in $\mathbb{H}$.

The problem of deciding whether a hypergraph is 2-colorable is called the HYPERGRAPH 2-COLORABILITY problem, and is known to be NP-hard [3]. Therefore, as a result of the above reduction, deciding whether $\mathbb{N}(G, S, T)$ has a routing solution is also NP-hard. ∎

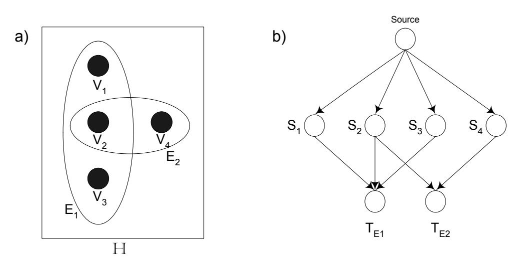

Fig. 3. a) A hypergraph $\mathbb{H}$. b) The corresponding 2-multicasting network resulting from the reduction of Theorem 9.

Now, we keep the minimality constraint on the graph G and we consider the case of multicasting $h > 2$ symbols.

Lemma 10: Let $\mathbb{N}(G, S, T)$ be an h-minimal coding network. If $\mathbb{N}(G, S, T)$ has a routing solution then the graph G'' is h-colorable.

Proof: Note that G'' is constructed here directly from G similarly to the case of $h = 2$ (see Section IV). The proof is then similar to the first part of the proof of Theorem 7. ∎

Lemma 10 leads to an algorithm for checking if $\mathbb{N}(G, S, T)$ does not have a routing solution. Such an algorithm would test if G'' is h-colorable. If it is not, it returns that there is no routing solution, otherwise it returns no answer. Since the problem of deciding if a graph is h-colorable is NP-hard in general, such an algorithm would not be efficient. However, the following theorem describes a family of h-minimal $\mathbb{N}(G, S, T)$ networks where this algorithm runs in linear time.

Theorem 11: Consider the problem of multicasting h symbols over an h-minimal network $\mathbb{N}(G, S, T)$, where every vertex of G of in-degree > 1 has at least 3 Steiner node parents. Then, there exists an algorithm that checks the non-existence of a routing solution of $\mathbb{N}(G, S, T)$ in $O(|V| + |E|)$.

Proof: Consider a vertex v in G of $d^-(v) > 1$ and having at least $\ell \geq 3$ Steiner node parents. The vertices in G'' corresponding to his parent nodes are pairwise connected. Thus v result in a K^ℓ (complete graph of ℓ vertices) subgraph of G''. Thus, by Proposition 5.5.1 of [1], we deduce that G'' is triangulated. By Lemma 10, an algorithm that would check the non-existence of a routing solution can be implemented in the following manner. First, it constructs G'' from G (can be done in $O(|E| + |V|)$ time). Then, it checks if G'' is h-colorable which can be done in $O(|V''| + |E''|)$ time, since G'' is triangulated. The total running time of the algorithm is $O(|E| + |V|)$. ■

Note that the converse of Lemma 10 is not always true. A counter example is provided in Figure 4 which shows a problem of multicasting 3 symbols. This problem does not have a routing solution in spite of G'' being 3-colorable.

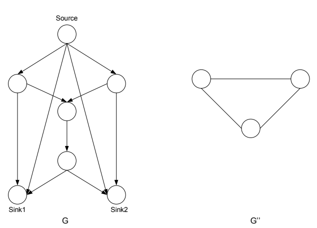

Fig. 4. A counter example to the converse of Lemma 10.

However, there are some instances of problems where the converse holds, as described by Theorem 12.

Theorem 12: If $\mathbb{N}(G, S, T)$ is defined over minimal graph G of vertices, except the source, of in-degree either 1 or h, then it has a routing solution if and only if the graph G'' is r-colorable.

Proof: (sketch)
($\Rightarrow$) same as Lemma 10.
($\Leftarrow$) Define the $M^h_{s,t}$ family of coding networks of capacity h, similarly to $M_{s,t}$ but connecting each sink to h intermediate nodes instead of just 2. By following the same steps of the proof of Theorem 2, it can be shown that $\mathbb{N}(G, S, T)$ is equivalent to some $\mathbb{N}'(G', S', T') \in M^h_{s,t}$. Then, as previously done in proving the converse of Theorem 7, by establishing a bijection between the h symbols and h colors, the coloring of G'' can be used to obtain a routing solution of $\mathbb{N}'$ and therefore $\mathbb{N}$. ■

Figure 5 depicts a 3-minimal coding network that satisfies the condition of Theorem 12.

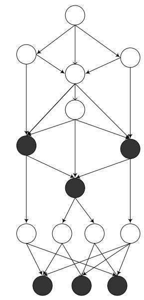

Fig. 5. An instance of $\mathbb{N}(G, S, T)$ where a necessary and sufficient condition for the existence of a routing solution is that G'' should be 3-colorable.

VI. Conclusion

In this paper, we have considered the network coding problem for a practically important class of multicast coding networks. We showed that minimal multicast networks have unique properties that distinguish them from the general case of multicast networks. Specifically, we proved that in such networks, when two symbols are to be communicated to the sinks ($h = 2$), the question whether the capacity can be achieved without network coding can be answered in polynomial time, while in general this problem is NP-hard. We also showed that when the network possesses some planarity property, a field of size 3 is sufficient for finding a feasible linear network code. In addition, we analyzed several structural properties of 2-minimal networks and generalized our results for $h > 2$.

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