ECE511: Analysis of Random Signals Fall 2016

Lecture Chapter 6: Convergence of Random Sequences

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1 Random sequence

Definition 1. *An infinite sequence X_n, $n = 1, 2, \dots,$ of random variables is called a random sequence.*

2 Convergence of a random sequence

Example 1. *Consider the sequence of real numbers*

$$X_n = \frac{n}{n+1},\ n = 0, 1, 2, \dots$$

This sequence converges to the limit $l = 1$. We write

$$\lim_{n\to\infty} X_n = l = 1.$$

This means that in any neighbourhood around 1 we can trap the sequence, i.e.,

$$\forall \epsilon > 0, \quad \exists\, n_0(\epsilon) \quad s.t. \quad for\ n \geq n_0(\epsilon) \quad |X_n - l| \leq \epsilon.$$

We can pick ϵ to be very small and make sure that the sequence will be trapped after reaching $n_0(\epsilon)$. Therefore as ϵ decreases $n_0(\epsilon)$ will increase. For example, in the considered sequence:

$$\epsilon = \frac{1}{2}, \qquad n_0(\epsilon) = 2,$$
$$\epsilon = \frac{1}{1000}, \quad n_0(\epsilon) = 1001.$$

2.1 Almost sure convergence

Definition 2. *A random sequence X_n, $n = 0, 1, 2, 3, \dots,$ converges* almost surely, or with probability one, *to the random variable X iff*

$$P(\lim_{n\to\infty} X_n = X) = 1.$$

We write

$$X_n \xrightarrow{a.s.} X.$$

Example 2. *Let ω be a random variable that is uniformly distributed on $[0,1]$. Define the random sequence X_n as $X_n = \omega^n$.*

$$\textit{So } X_0 = 1,\ X_1 = \omega,\ X_2 = \omega^2,\ \ X_3 = \omega^3, \ldots$$

Let us take specific values of ω. For instance, if $\omega = \frac{1}{2}$

$$X_0 = 1,\ X_1 = \frac{1}{2},\ X_2 = \frac{1}{4}\ \ ,\ X_3 = \frac{1}{8}, \ldots$$

We can think of it as an urn containing sequences, and at each time we draw a value of ω, we get a sequence of fixed numbers. In the example of tossing a coin, the output will be either heads or tails. Whereas, in this case the output of the experiment is a random sequence, i.e., each outcome is a sequence of infinite numbers.

Question: Does this sequence of random variables converge?

Answer: This sequence converges to

$$X = \begin{cases} 0 & \text{if } \omega \neq 1 \text{ with probability } 1 = P\left(\omega \neq 1\right) \\ 1 & \text{if } \omega = 1 \text{ with probability } 0 = P\left(\omega = 1\right) \end{cases}$$

Since the pdf is continuous, the probability $P(\omega = a) = 0$ for any constant a. Notice that the convergence of the sequence to 1 is possible but happens with probability 0.

Therefore, we say that X_n converges almost surely to 0, i.e., $X_n \xrightarrow{a.s.} 0$.

2.2 Convergence in probability

Definition 3. *A random sequence X_n converges to the random variable X in probability if*

$$\forall \epsilon > 0 \quad \lim_{n\to\infty} Pr\left\{|X_n - X| \geq \epsilon\right\} = 0.$$

We write :

$$X_n \xrightarrow{p} X.$$

Example 3. *Consider a random variable ω uniformly distributed on $[0,1]$ and the sequence X_n defined by:*

$$X_n = \begin{cases} 0 & \textit{with probability } \frac{\omega}{n} \\ 1 & \textit{with probability } 1 - \frac{\omega}{n} \end{cases}$$

Distributed as shown in Figure 1. Notice that only X_2 or X_3 can be equal to 1 for the same value of ω. Similarly, only one of X_4, X_5, X_6 and X_7 can be equal to 1 for the same value of ω and so on and so forth.

Question: Does this sequence converge?

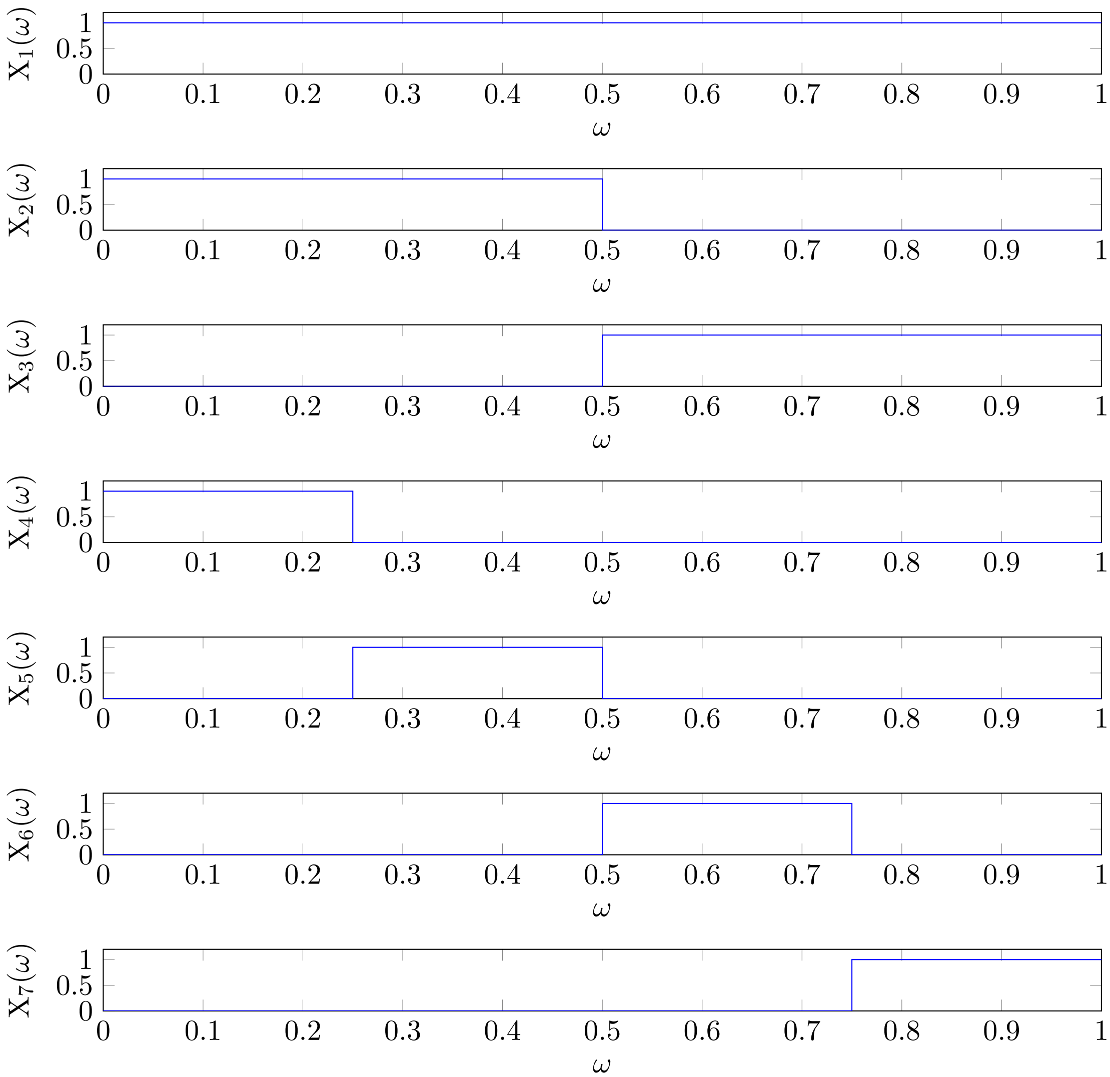


Figure 1: Plot of the distribution of $X_n(\omega)$

Answer: Intuitively, the sequence will converge to 0. Let us take some examples to see how the sequence behave.

$$\text{for } \omega = 0 : \quad \underbrace{1}_{n=1}\underbrace{10}_{n=2}\underbrace{1000}_{n=3}\underbrace{10000000}_{n=4}\dots$$

$$\text{for } \omega = \frac{1}{3} : \quad \underbrace{1}_{n=1}\underbrace{10}_{n=2}\underbrace{0100}_{n=3}\underbrace{00100000}_{n=4}\dots$$

From a calculus point of view, these sequences never converge to zero because there is always a "jump" showing up no matter how many zeros are preceding (Fig. 2); for any ω : $X_n(\omega)$ does not converge in the "calculus" sense. Which means also that X_n does not converge to zero almost surely (a.s.).

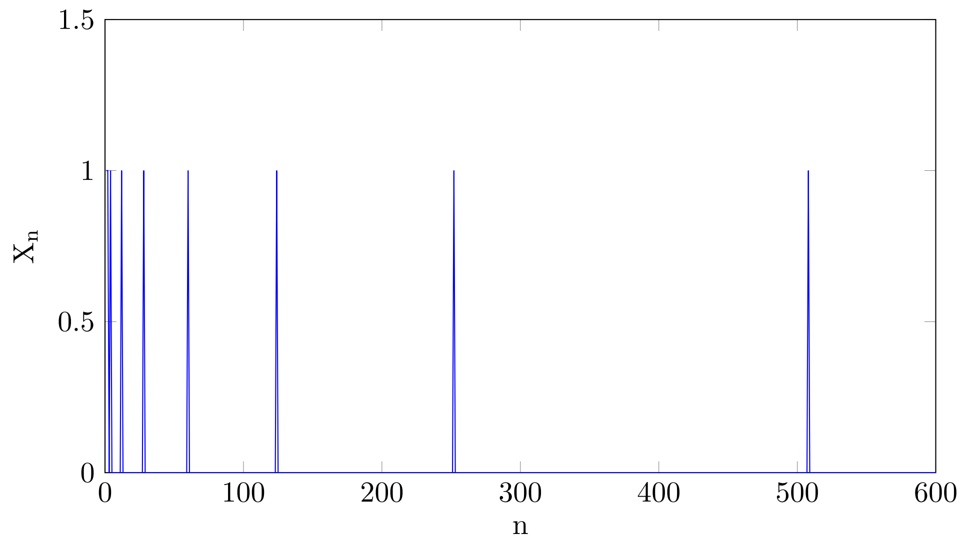


Figure 2: Plot of the sequence for $\omega = 0$

This sequence converges in probability since

$$\lim_{n\to\infty} P\left(|X_n - 0| \geq 0\right) = 0 \quad \forall \epsilon > 0.$$

Remark 1. *The observed sequence may not converge in "calculus" sense because of the intermittent "jumps"; however the frequency of those "jumps"* goes *to zero when n goes to infinity.*

2.3 Convergence in mean square

Definition 4. *A random sequence X_n converges to a random variable X in mean square sense if*

$$\lim_{n\to\infty} E\left[|X - X_n|^2\right] = 0.$$

We write:

$$X_n \xrightarrow{m.s.} X.$$

Remark 2. *In mean square convergence, not only the frequency of the "jumps" goes to zero when n goes to infinity; but also the "energy" in the jump should go to zero.*

Example 4. *Consider a random variable ω uniformly distributed over $[0,1]$, and the sequence $X_n(\omega)$ defined as:*

$$X_n(\omega) = \begin{cases} a_n & \text{for } \omega \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Note that $P\left(X_n = a_n\right) = \frac{1}{n}$ and $P\left(X_n = 0\right) = 1 - \frac{1}{n}$.

Question: Does this sequence converge?

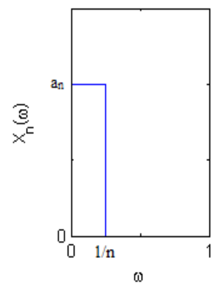


Figure 3: Plot of the sequence $X_n(\omega)$

Answer: Let us check the different convergence criteria we have see so far.

1. *Almost sure convergence:* $X_n \xrightarrow{a.s.} 0$ because
$$\lim_{n\to\infty} P(X_n = 0) = 1.$$

2. *Convergence in probability:* $X_n \xrightarrow{p.} 0$ because
$$\lim_{n\to\infty} P\left\{|X_n - 0| \le \epsilon\right\} = 0.$$
(Flash Forward: almost sure convergence $\Rightarrow$ convergence in probability.)
$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p.} X.$$

3. *Mean Square Convergence:*
$$\begin{aligned} E\left[|X_n - 0|^2\right] &= a_n^2\left(P\left(X_n = a_n\right) + 0P\left(X_n = 0\right)\right), \\ &= \frac{a_n^2}{n}. \end{aligned}$$
$$\begin{aligned} \text{If } a_n = 10 &\Rightarrow \lim_{n\to\infty} E\left[|X_n - 0|^2\right] = 0 \Rightarrow X_n \xrightarrow{m.s.} 0, \\ \text{If } a_n = \sqrt{n} &\Rightarrow \lim_{n\to\infty} E\left[|X_n - 0|^2\right] = 1 \Rightarrow X_n \text{ does not converge in m.s. to 0.} \end{aligned}$$
In this example, the convergence of X_n in the mean square sense depends on the value of a_n.

2.4 Convergence in distribution

Definition 5. *(First attempt) A random sequence X_n converges to X in distribution if when n goes to infinity, the values of the sequence are distributed according to a known distribution. We say*
$$X_n \xrightarrow{d.} X.$$

Example 5. *Consider the sequence X_n defined as:*
$$X_n = \begin{cases} X_i \sim B(\frac{1}{2}) & \textit{for } i = 1 \\ (X_{i-1} + 1) \mod 2 = X \oplus 1 & \textit{for } i > 1 \end{cases}$$

Question: In which sense, if any, does this sequence converge?

Answer: This sequence has two outcomes depending on the value of X_1:

$$X_1 = 1, \quad X_n : 101010101010\ldots$$
$$X_1 = 0, \quad X_n : 010101010101\ldots$$

1. *Almost sure convergence:* X_n does not converge almost surely because the probability of every jump is always equal to $\frac{1}{2}$.
2. *Convergence in probability:* X_n does not converge in probability because the frequency of the jumps is constant equal to $\frac{1}{2}$.
3. *Convergence in mean square:* X_n does not converge to $\frac{1}{2}$ in mean square sense because

$$\begin{aligned}\lim_{n\to\infty} E\left[|X_n - \frac{1}{2}|^2\right] &= E\left[X_n^2 - X_n + \frac{1}{4}\right], \\ &= E[X_n^2] - E[X_n] + \frac{1}{4}, \\ &= 1^2\frac{1}{2} + 0^2\frac{1}{2} - 0 + \frac{1}{4}, \\ &= \frac{1}{2}.\end{aligned}$$

4. *Convergence in distribution:* At infinity, since we do not know the value of X_1, each value of X_n can be either 0 or 1 with probability $\frac{1}{2}$. Hence, any number X_n is a random variable $\sim B(\frac{1}{2})$. We say, X_n converges in distribution to Bernoulli($\frac{1}{2}$) and we denote it by:

$$X_n \xrightarrow{d} Ber(\frac{1}{2}).$$

Example 6. *(Central Limit Theorem)Consider the zero-mean, unit-variance, independent random variables $X_1, X_2, \ldots, X_n$ and define the sequence S_n as follows:*

$$S_n = \frac{X_1 + X_2 + ... + X_n}{\sqrt{n}}.$$

The CLT states that S_n converges in distribution to $N(0,1)$, i.e.,

$$S_n \xrightarrow{d} N(0,1).$$

Theorem 1.

$$\left.\begin{array}{l}\textit{Almost sure convergence}\\ \textit{Convergence in mean square}\end{array}\right\} \Rightarrow \textit{Convergence in probability} \Rightarrow \textit{convergence in distribution.}$$

Note:

- *There is no relation between Almost Sure and Mean Square Convergence.*
- *The relation is unidirectional, i.e., convergence in distribution does not imply convergence in probability neither almost sure convergence nor mean square convergence.*

3 Convergence of a random sequence

Example 1: Let the random variable U be uniformly distributed on $[0,1]$. Consider the sequence defined as:

$$X(n) = \frac{(-1)^n U}{n}.$$

Question: Does this sequence converge? if yes, in what sense(s)?

Answer:

1. *Almost sure convergence:* Suppose

$$U = a.$$

The sequence becomes

$$\begin{aligned} X_1 &= -a, \\ X_2 &= \frac{a}{2}, \\ X_3 &= -\frac{a}{3}, \\ X_4 &= \frac{a}{4}, \\ &\vdots \end{aligned}$$

In fact, for any $a \in [0,1]$

$$\lim_{n\to\infty} X_n = 0,$$

therefore, $X_n \xrightarrow{a.s.} 0$.

Remark 3. $X_n \xrightarrow{a.s.} 0$ *because, by definition, a random sequence converges almost surely to the random variable* X *if the sequence of functions* X_n *converges for all values of* U *except for a set of values that has a probability zero.*

2. *Convergence in probability:* Does $X_n \xrightarrow{p.} 0$? Recall from theorem 13 of lecture 17:

$$\left.\begin{matrix} \text{a.s.} \\ \text{m.s.} \end{matrix}\right\} \Rightarrow \text{p.} \Rightarrow \text{d.}$$

which means that by proving almost-sure convergence, we get directly the convergence in probability and in distribution. However, for completeness we will formally prove that X_n converges to 0 in probability. To do so, we have to prove that

$$\lim_{n\to\infty} P\left(|X - 0| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0,$$
$$\Rightarrow \lim_{n\to\infty} P\left(|X_n| \geq \epsilon\right) = 0 \quad \forall \epsilon > 0.$$

By definition,

$$|X_n| = \frac{U}{n} \leq \frac{1}{n}.$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} P\Big(|X_n| \geq \epsilon\Big) &= \lim_{n\to\infty} P\left(\frac{U}{n} \geq \epsilon\right), && (1)\\
&= \lim_{n\to\infty} P\left(U \geq n\epsilon\right), && (2)\\
&= 0. && (3)
\end{aligned}$$

Where equation 3 follows from the fact that finding $U \in [0, 1]$.

3. *Convergence in mean square sense:* Does X_n converge to 0 in the mean square sense?

   In order to answer this question, we need to prove that

$$\lim_{n\to\infty} E\left[|X_n - 0|^2\right] = 0.$$

   We know that,

$$\begin{aligned}
\lim_{n\to\infty} E\left[|X_n - 0|^2\right] &= \lim_{n\to\infty} E\left[X_n^2\right],\\
&= \lim_{n\to\infty} E\left[\frac{U^2}{n^2}\right],\\
&= \lim_{n\to\infty} \frac{1}{n^2} E\left[U^2\right],\\
&= \lim_{n\to\infty} \frac{1}{n^2} \int_0^1 u^2 du,\\
&= \lim_{n\to\infty} \frac{1}{n^2} \left.\frac{u^3}{3}\right]_0^1,\\
&= \lim_{n\to\infty} \frac{1}{3n^2},\\
&= 0.
\end{aligned}$$

   Hence, $X_n \xrightarrow{m.s.} 0$.

4. *Convergence in distribution:* Does X_n converge to 0 in distribution? The formal definition of convergence in distribution is the following:

$$X_n \xrightarrow{d.} X \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x).$$

   Hereafter, we want to prove that $X_n \xrightarrow{d.} 0$.

   Recall that the limit r.v. X is the constant 0 and therefore has the following CDF :
   Since $X_n = \frac{(-1)^n U}{n}$, the distribution of the X_i can be derived as following:

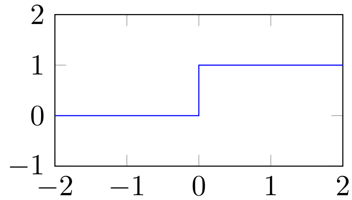
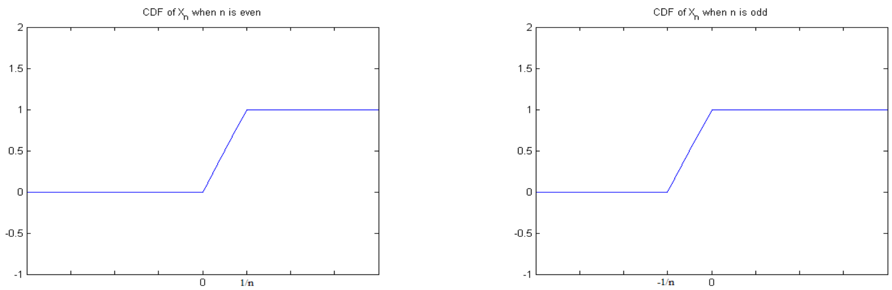


Figure 4: Plot of the CDF of 0



Remark 4. *At 0 the CDF of X_n will be flip-flopping between 0 (if n is even) and 1 (if n is odd) (c.f. figure 5) which implies that there is a discontinuity at that point. Therefore, we say that X_n converges in distribution to a CDF $F_X(x)$ except at points where $F_X(x)$ is not continuous.*

Definition 6. *X_n converges to X in distribution, i.e., $X[n] \xrightarrow{d.} X$ iff*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \text{except at points where } F_X(x) \text{ is not continuous.}$$

Remark 5. *It is clear here that*

$$\lim_{n\to\infty} F_{X_n}(x) = F_x(x) \quad \text{except for } x = 0.$$

Therefore, X_n converges to X in distribution. We could have deduced this directly from convergence in mean square sense or almost sure convergence.

Theorem 2. *a) If $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{p.} X$.*

b) If $X_n \xrightarrow{m.s.} X \Rightarrow X_n \xrightarrow{p.} X$.

c) If $X_n \xrightarrow{p.} X \Rightarrow X_n \xrightarrow{d.} X$.

d) If $P\{|X_n| \leq Y\} = 1$ for all n for a random variable Y with $E\left[Y^2\right] < \infty$, then

$$X_n \xrightarrow{p.} X \Rightarrow X_n \xrightarrow{m.s.} X.$$

.

Proof. The proof is omitted. □

Remark 6. *Convergence in probability allows the sequence, at ∞, to deviate from the mean for any value with a small probability; whereas, convergence in mean square limits the amplitude of this deviation when $n \to \infty$. (We can think of it as energy $\Rightarrow$ we can not allow a big deviation from the mean).*

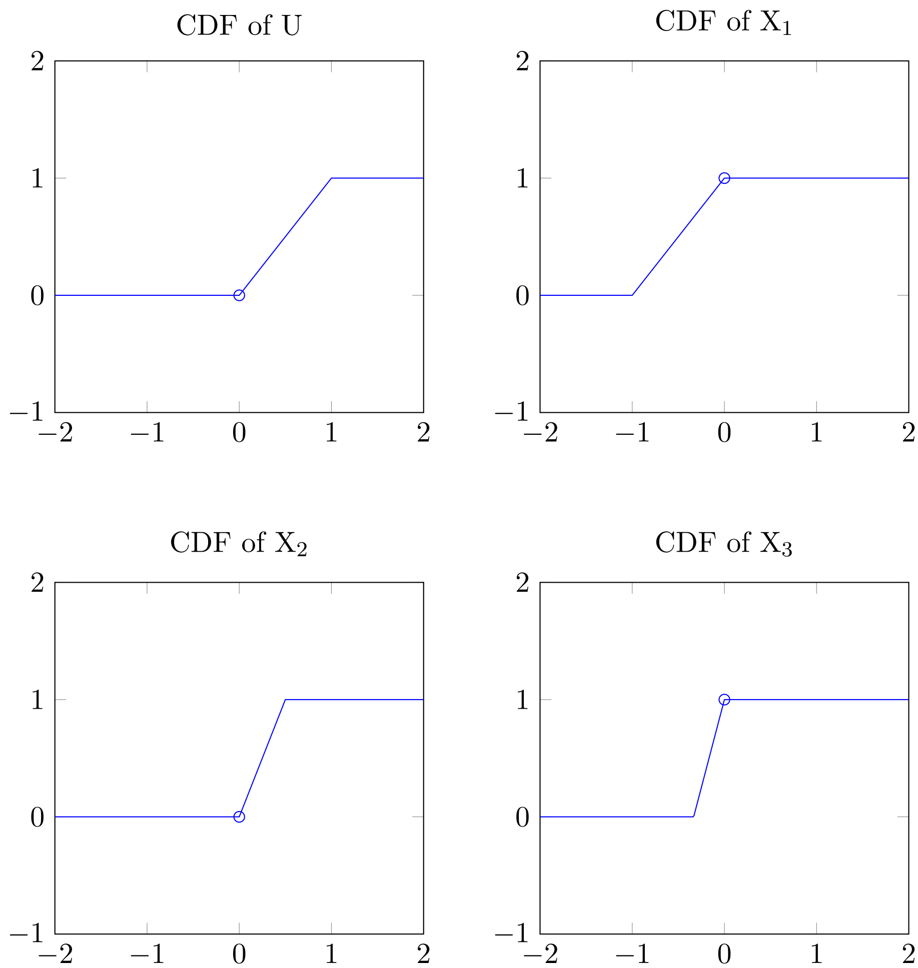


Figure 5: Plot of the CDF of U, X_1, X_2 and X_3

4 Back to real analysis

Definition 7. *A sequence $(x_n)_{n\geq 1}$ is Cauchy if for every ϵ, there exists a large number N s.t.*

$$\forall\, m, n > N,\ \ |x_m - x_n| < \epsilon \quad \Leftrightarrow \quad \lim_{n,m\to\infty} |x_m - x_n| = 0.$$

Claim 1. *Every Cauchy sequence is convergent.*

Counter example 1. *Consider the sequence $X_n \in \mathbb{Q}$ defined as $x_0 = 1$, $x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}$. The limit of this sequence is given by:*

$$\begin{aligned} l &= \frac{l + \frac{2}{l}}{2}, \\ 2l^2 &= l^2 + 2, \\ l &= \pm\sqrt{2} \notin \mathbb{Q}. \end{aligned}$$

This implies that the sequence does not converge in $\mathbb{Q}$.

Counter example 2. *Consider the sequence $x_n = 1/n$ in $(0,1)$. Obviously it does not converge in $(0,1)$ since the limit $l = 1 \notin (0,1)$.*

Definition 8. *A space where every sequence converges is called a complete space.*

Theorem 3. *$\mathbb{R}$ is a complete space.*

Proof. The proof is omitted. □

Theorem 4. *Cauchy criteria for convergence of a random sequence.*

a) $X_n \xrightarrow{a.s.} X \iff P\left[\lim_{m,n\to\infty} |x_m - x_n| = 0\right] = 1.$

b) $X_n \xrightarrow{m.s.} X \iff \lim_{m,n\to\infty} E\left[|x_m - x_n|^2\right] = 0.$

c) $X_n \xrightarrow{p.} X \iff \lim_{m,n\to\infty} P\left[|x_m - x_n| \geq \varepsilon\right] = 0 \quad \forall \epsilon.$

Proof. The proofs are omitted. □

Example 7. *Consider the sequence of example 11 from last lecture,*

$$X_n = \begin{cases} X_i \sim B(\frac{1}{2}) & \text{for } i = 1 \\ (X_{i-1}+1) \mod 2 = X \oplus 1 & \text{for } i > 1 \end{cases}$$

Goal: Our goal is to prove that this sequence does not converge in mean square using Cauchy criteria.

This sequence has two outcomes depending on the value of X_1:

$$\begin{aligned} X_1 = 1, &\quad X_n : 101010101010\ldots \\ X_1 = 0, &\quad X_n : 010101010101\ldots \end{aligned}$$

Therefore,

$$\begin{aligned} E\left[|X_n - X_m|^2\right] &= E\left[X_n^2\right] + E\left[X_m^2\right] - 2E\left[X_m X_n\right], \\ &= \frac{1}{2} + \frac{1}{2} - 2E\left[X_m X_n\right]. \end{aligned}$$

Consider, without loss of generality, that $m > n$

$$E\left[X_n X_m\right] = \begin{cases} E\left[X_n X_m\right] = 0 & \text{if } m - n \text{ is odd,} \\ E\left[X_n^2\right] = \frac{1}{2} & \text{if } m - n \text{ is even.} \end{cases}$$

Hence,

$$\lim_{n,m\to\infty} E\left[|X_n - X_m|^2\right] = \begin{cases} 1 & \text{if } m - n \text{ is odd,} \\ 0 & \text{if } m - n \text{ is even,} \end{cases}$$

which implies that X_n does not converge in mean square by theorem 4-*b*).

Lemma 1. *Let X_n be a random sequence with $E\left[X_n^2\right] < \infty\ \forall n$.*

$$X_n \xrightarrow{m.s.} X \quad \textit{iff} \quad \lim_{m,n\to\infty} E\left[X_m X_n\right] \ \textit{exists and is finite.}$$

Theorem 5. *Weak law of large numbers*

Let $X_1, X_2, X_3, \ldots, X_i$ be iid random variables. $E\left[X_i\right] = \mu$, $\forall i$. Let

$$S_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

Then

$$P\left[|S_n - \mu| \geq \epsilon\right] \xrightarrow[n\to\infty]{} 0.$$

Using the language of this chapter:

$$S_n \xrightarrow{p.} \mu.$$

Theorem 6. *Strong law of large numbers*

Let $X_1, X_2, X_3, \ldots, X_i$ be iid random variables. $E\left[X_i\right] = \mu$, $\forall i$. Let

$$S_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

Then

$$P\left[\lim_{n\to\infty} |S_n - \mu| \geq \epsilon\right] = 0.$$

Using the language of this chapter:

$$S_n \xrightarrow{a.s.} \mu.$$

Theorem 7. *Central limit theorem*

Let $X_1, X_2, X_3, \ldots, X_i$ be iid random variables. $E\left[X_i\right] = 0$, $\forall i$. Let

$$Z_n = \frac{X_1 + X_2 + ... + X_n}{\sqrt{n}}.$$

Then

$$P\left[Z_n \leq z\right] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

Using the language of this chapter:

$$Z_n \xrightarrow{d.} N(0,1).$$