ECE511: Analysis of Random Signals Fall 2016

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Chapter 2: Random Variables

Example 1. *Tossing a fair coin twice:*

$$\Omega = \{HH, HT, TH, TT\}.$$

Define for any $\omega \in \Omega$, $X(\omega)$=number of heads in ω. $X(\omega)$ is a random variable.

Definition 1. *A random variable (RV) is a function X: $\Omega \to R$.*

Definition 2 (Cumulative distribution function(CDF))**.**

$$F(\chi) = P(X \leq \chi). \tag{1}$$

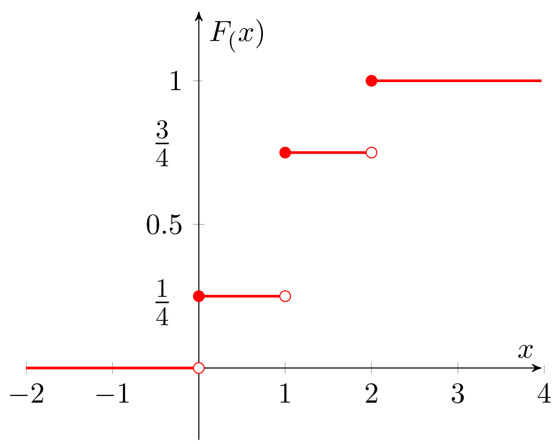

Figure 1: Cumulative distribution function of x

Example 2. *The cumulative distribution function of x is as (Figure 1)*

$$F_X(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{4} & 0 \leq x < 1, \\ \frac{3}{4} & 1 \leq x < 2, \\ 1 & x \geq 2. \end{cases}$$

Lemma 1. *Properties of CDF*

(1)

$$\lim_{x \to -\infty} F_X(x) = 0 \tag{2}$$

$$\lim_{x \to +\infty} F_X(x) = 1, \tag{3}$$

(2) $F_X(x)$ is non-decreasing:

$$x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2) \tag{4}$$

(3) $F_X(x)$is continuous from the right

$$\lim_{\epsilon \to 0} F_X(x+\epsilon) = F_X(x), \epsilon > 0 \tag{5}$$

(4)

$$\begin{aligned} P(a \leq X \leq b) &= P(X \leq b) - P(X \leq a) + P(X = a) && (6) \\ &= F_X(b) - F_X(a) + P(X = a) && (7) \end{aligned}$$

(5)

$$P(X = a) = \lim_{\varepsilon \to 0} F_X(a) - F_X(a - \varepsilon), \varepsilon > 0 \tag{8}$$

Definition 3. *If random variable X has finite or countable number of values, X is called discrete.*

Example 3. *Non-countable example:$\Re$.*
A set S is countable if you can find a bijection of f.

$$f : S \to N.$$

Definition 4. *X is continuous if $F_X(x)$ is continuous.*

Definition 5 (Probability density function(pdf))**.**

$$f_X(x) \quad = \quad \frac{dF_X(x)}{dx} \ (x \text{ is continuous}). \tag{9}$$

Example 4. *Gaussian random variable: Normal/ Gaussian Distribution.*

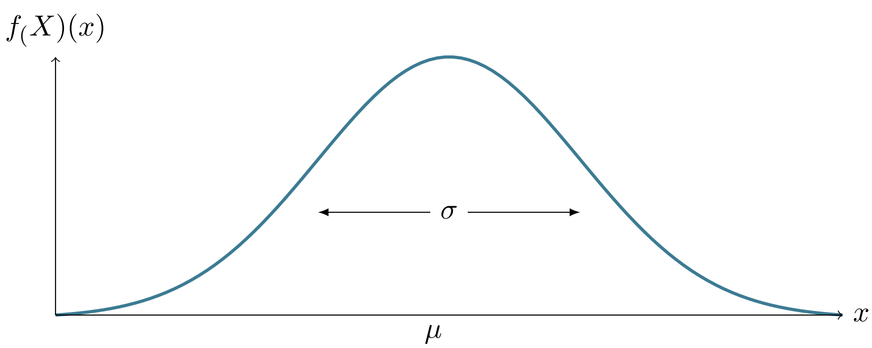

Figure 2: Gaussian distribution pdf.

By definition,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}}$$

Therefore,

$$\begin{aligned} F_X(a) &= P(x \leq a) = \int_{-\infty}^{a} f_X(x)dx, \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{a} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}} dx. \end{aligned}$$

We should always have:

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1.$$

Definition 6 (mean, variance of a RV X)**.** *For the continuous case:*

$$\begin{aligned} E(X) = \mu &= \int_{-\infty}^{+\infty} x f_X(x)dx, \\ V(X) = \sigma^2 &= \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)dx. \end{aligned}$$

For the discrete case:

$$\begin{aligned} E(X) &= \mu = \sum_{i=-\infty}^{+\infty} x_i P(X = x_i), \\ V(X) &= \sigma^2 = \sum_{i=-\infty}^{+\infty} (x_i - \mu)^2 P(X = x_i). \end{aligned}$$

Example 5. *X is uniformly distributed in* $(0, 1]$.

$$F_X(x) = \begin{cases} 0 & x < 0, \\ \int_0^x 1dx = x & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

$$\begin{aligned} E(X) &= \int_0^1 X \times 1dx = \frac{1}{2}, \\ V(X) &= \int_0^1 (X - \frac{1}{2})^2 \times 1dx = \frac{1}{12}. \end{aligned}$$

Lemma 2 (Probability Density Functions)**.**

(1) Uniform X uniform over [a,b]:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \textit{if } a \leq x \leq b \\ 0 & \textit{otherwise} \end{cases} \tag{10}$$

[1] Figure from Wikipedia: `https://en.wikipedia.org/wiki/Uniform_distribution_(continuous)`

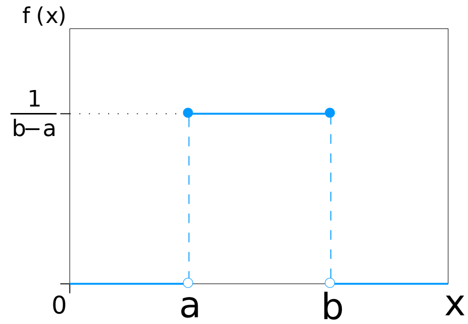

Figure 3: Uniform distribution.[1]

(2) Gaussian distribution:

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}}, \tag{11}$$

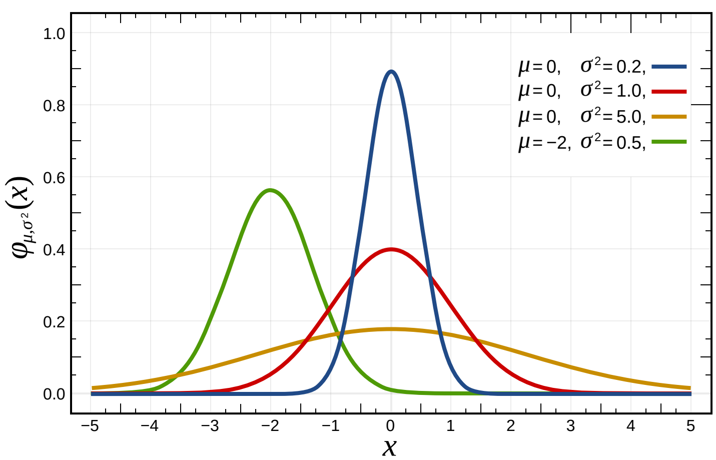

Figure 4: Gaussian distribution.[2]

(3) Exponential distribution:

$$f_X(x) = \begin{cases} \frac{1}{\mu} e^{\frac{-x}{\mu}} & \textit{if } x \geq 0 \\ 0 & \textit{if } x < 0 \end{cases} \tag{12}$$

(4) Rayleigh Distribution:

$$f_X(x) \quad = \quad \frac{x}{\sigma^2} e^{\dfrac{-x^2}{2\sigma^2}}, x \geq 0, \tag{13}$$

[2]Figure from Wikipedia: `https://en.wikipedia.org/wiki/Normal_distribution`
[3]Figure from Wikipedia: `https://en.wikipedia.org/wiki/Exponential_distribution`
[4]Figure from Wikipedia: `https://en.wikipedia.org/wiki/Rayleigh_distribution`

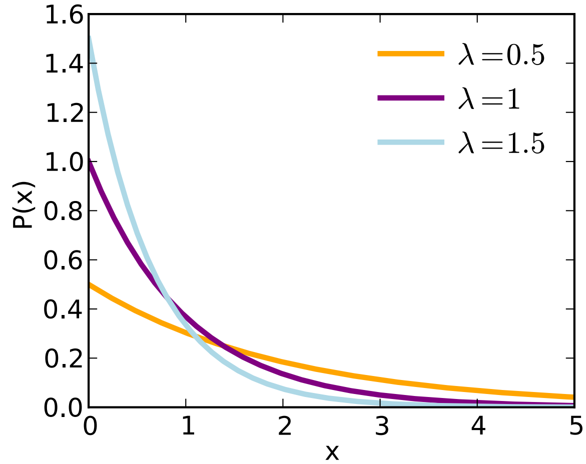

Figure 5: Exponential distribution.[3]

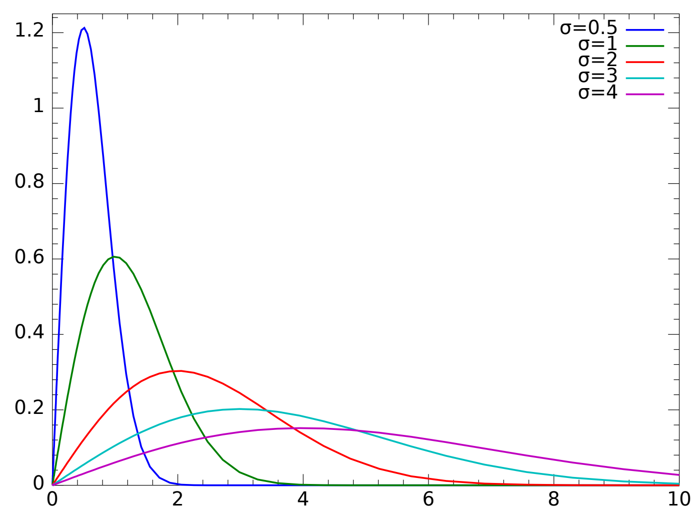

Figure 6: Rayleigh distribution.[4]

(5) Laplacian Distribution:

$$f_X(x) = \frac{1}{\sqrt{2}\sigma} e^{\frac{-\sqrt{2}|x|}{\sigma}}. \tag{14}$$

[5]Figure from Wikipedia: `https://en.wikipedia.org/wiki/Laplace_distribution`

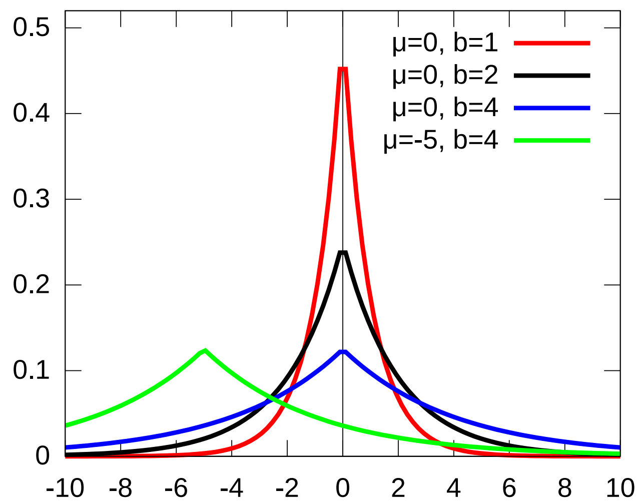

Figure 7: Laplacian distribution.[5]

1 Example of Discrete Random Variable

1.1 Bernoulli RV

flipping a coin, $P(H) = p$, $P(T) = 1 - p$, if head occurs $X = 1$, if tail occurs $X = 0$, $P(X = 0) = 1 - p, P(X = 1) = p$. The CDF of a bernoulli RV is as Figure 8.

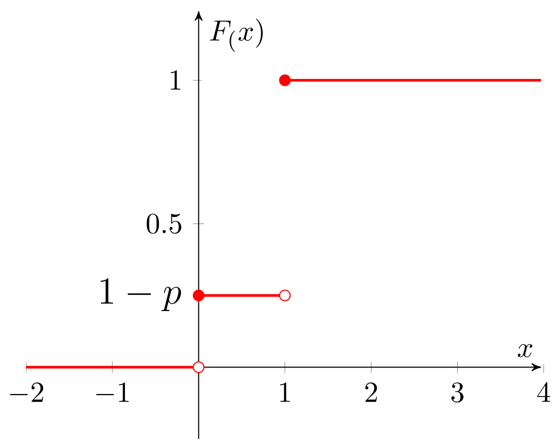

Figure 8: Cumulative distribution function of Bernoulli Random Variable

1.2 Binomial distribution

Tossing a die n times, $P(H) = p$, $P(T) = 1 - p$. X is number of heads, $x \in \{0, 1, \ldots, n\}$.

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

Remark 1. *Let $Y_i \in \{0, 1\}$ denote the outcome of tossing the die the ith time*

$$X = Y_1 + Y_2 + \cdots + Y_n.$$

i.e., a Binomial RV can be thought of as the sum of n independent Bernoulli RV.

Example 6 (Random graph)**.** *Each edge exists with probability p, X is the number of neighbor of node 1(Figure 9).*

$$Y_i = \begin{cases} 1, & \textit{if node 1 is connected to i+1}, \\ 0, & \textit{otherwise.} \end{cases}$$

$$X = Y_1 + Y_2 + \cdots + Y_{n-1}.$$

So X follows the Binomial distribution.

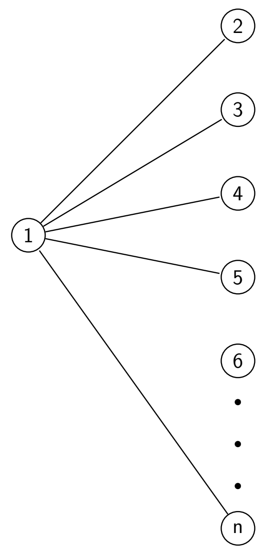

Figure 9: Random Graphs

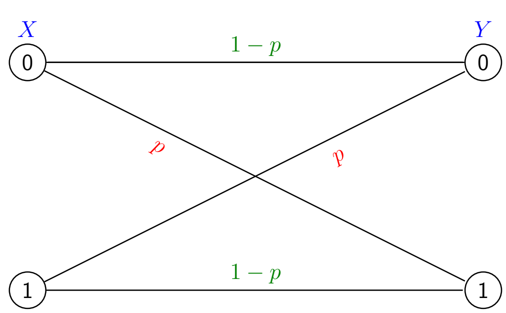

Figure 10: Binary Symmetric Channel with probability of error $P_e = p$.

Example 7 (BSC)**.** *Suppose we are transmitting a file of length n. Consider a BSC where the probability of error is p and the probability of receiving the correct bit is 1-p. (Figure 10) What is* *the probability that we have k errors?*

$$P(k\ errors) = \binom{n}{k} p^k (1-p)^{n-k}$$

Let X represent the number of errors, what is $E(X)$

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} kP(X=k), \\ &= \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k}, \\ &= np\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}, \\ &= np\sum_{k=0}^{n-1} \binom{n-1}{k} p^{k} (1-p)^{n-k+1}, \\ &= np. \end{aligned}$$

Binomial theorem:

$$\begin{aligned} (x+y)^n &= \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k} \\ (p+1-p)^{n-1} &= \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k+1} \\ &= 1. \end{aligned}$$

Theorem 1. *For any two RVs X_1 and X_2, $Y = X_1 + X_2$,*

$$E(Y) = E(X_1) + E(X_2). \tag{15}$$

It does not matter whether X_1 and X_2 are independent or not.

1.3 Geometric distribution

You keep tossing a coin until you observe a Head. X is the number of times you have to toss the coin.

$$\begin{aligned} X &\in \{1, 2, \dots\}, \\ P(X = K) &= (1-p)^k p. \end{aligned}$$

Example 8 (Binary erasure channel)**.** *Suppose you have a BEC channel with feedback. When you get a erasure, you ask the sender to retransmit.(Figure 11)*
Suppose you pay one dollar for each retransmission. Let X be the amount of money you pay per transmission.

$$\begin{aligned} E(X) &= \frac{1}{1-p}, \\ &= \frac{1}{0.9} \approx 1.11\$. \end{aligned}$$

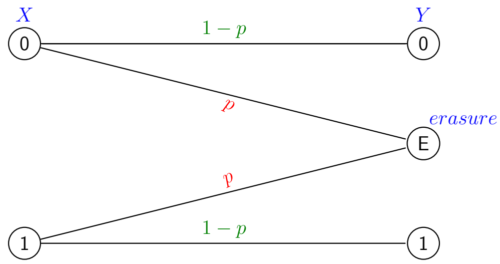

Figure 11: Binary Erasure Channel with probability of erasure $P_e = 0.1$.

For geometric distribution,

$$P(H) \approx \frac{1}{E(X)},$$

which $E(X)$is the number of coin flips on average.

Proof.

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X=k), && (16) \\ &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p, && (17) \\ &= p\sum_{k=1}^{\infty} k(1-p)^{k-1}. && (18) \end{aligned}$$

Recall that for $|x| < 1$,

$$\begin{aligned} \sum_{k=0}^{\infty} x^k &= \frac{1}{1-x}, && (19) \\ \frac{d}{dk}\sum_{k=1}^{\infty} kx^{k-1} &= \frac{1}{(1-x)^2}, && (20) \\ \sum_{k=1}^{\infty} k(1-p)^k &= \frac{1}{p^2}. && (21) \end{aligned}$$

So,

$$\begin{aligned} E(X) &= p\frac{1}{p^2}, && (22) \\ &= \frac{1}{p}. && (23) \end{aligned}$$

□

1.4 Poisson distribution

Suppose a server receives λ searches per second on average. The probability that the server receives k searches for this second is

$$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda},\ k=0,1,2,\ldots,\infty. \tag{24}$$

For poisson distribution

$$\begin{aligned}
\sum_{k=0}^{\infty} P(X=k) &= \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda} && (25)\\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} && (26)\\
&= 1. && (27)
\end{aligned}$$

$$E(X) = \lambda. \tag{28}$$

Example 9 (Interpretation of poisson distribution as an arrival experiment)**.**
suppose average of arrival cumtomers per second is λ. Suppose server goes down if $X \geq 100$. We want to find the probability of $P(X=k)$.

$$P(\textit{server going down}) = P(X \geq 100).$$

We divide the one second to n intervals, each length of the interval is $\frac{1}{n}$ second. The probability p of getting requests in small interval is $\frac{\lambda}{n}$.

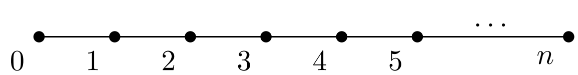

Figure 12: one second divided into n intervals.

Now we can consider it to be Bernoulli distribution with p.

$$\begin{aligned}
P(X=k) &= \binom{n}{k}p^k(1-p)^{n-k}, && (29)\\
&= \binom{n}{k}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} && (30)\\
&\approx \frac{n^k}{k!}(\frac{p}{1-p})^k(1-p)^n, && (31)\\
&= \frac{1}{k!}(np)^k e^{-np}, && (32)\\
&= \frac{1}{k!}\lambda^k e^{-\lambda}, && (33)\\
&= \frac{\lambda^k}{k!}e^{-\lambda}. && (34)
\end{aligned}$$

We get (31) *because of*

$$\binom{n}{k} = \frac{1}{k!}n(n-1)\dots(n-k+1), \tag{35}$$

$$\approx \frac{n^k}{k!},\ (k \textit{ is a constant and } n \textit{ goes to infinity}). \tag{36}$$

This means we can approximate Binomial(n,p) by Poisson with $\lambda = np$ *(if* n *is very large).*

2 Two Random Variables

Example 10. *Let* X *and* Y *be Bernoulli Random Variable. If* $Y = 0$*, we know* X *must equal to*

	Y=0	Y=1
X=0	$\frac{1}{2}$	$\frac{1}{4}$
X=1	0	$\frac{1}{4}$

Table 1: Joint probability mass function of X and Y.

0*, so* X *and* Y *are dependent.*

$$P(X = 0) = \frac{3}{4},$$

$$P(X = 1) = \frac{1}{4}.$$

Here is a example which X *and* Y *are independent, but they have the same marginal distribution.*

	Y=0	Y=1
X=0	$\frac{3}{8}$	$\frac{3}{8}$
X=1	$\frac{1}{8}$	$\frac{1}{8}$

Table 2: Joint probability mass function of X and Y.

2.1 Marginalization

You have the joint distribution $P_{X,Y}(x, y)$.

$$P_X(x_0) = \sum_y P_{X,Y}(x_0, y), \tag{37}$$

$$P_Y(y_0) = \sum_x P_{X,Y}(x, y_0). \tag{38}$$

Definition 7. *If* X *and* Y *are continuous random variables, then the joint CDF:*

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y). \tag{39}$$

Given joint CDF $F_{X,Y}(x, y)$,

$$F_X(x_0) = F_{X,Y}(x_0, +\infty). \tag{40}$$

Definition 8. *When the CDF is differentiable, the joint pdf is defined as*

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}, \tag{41}$$

$$f_X(x) = \int_{\infty}^{-\infty} f_{X,Y}(x,y)dy, \tag{42}$$

$$f_Y(y) = \int_{\infty}^{-\infty} f_{X,Y}(x,y)dx. \tag{43}$$

Definition 9. *X and Y are independent if and only if*

$$F_{X,Y}(x,y) = F_X(x)F_Y(y), \tag{44}$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y). \tag{45}$$

Definition 10. *Conditional CDF of marginal distribution is*

$$F_{X,Y}(x|y) = P(X \leq x | Y \leq y), \tag{46}$$

$$= \frac{F_{X,Y}(X \leq x, Y \leq y)}{P(Y \leq y)}. \tag{47}$$

Example 11. *X and Y are 2 random variables given by the joint pdf*

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} exp[\frac{-1}{2\sigma^2(1-\rho^2)}(x^2+y^2-2\rho xy)].$$

What is $f_X(x)$?

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \\
&= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} exp[\frac{-1}{2\sigma^2(1-\rho^2)}(x^2+y^2-2\rho xy)]dy, \\
&= \frac{exp[\frac{-x^2}{2\sigma^2(1-\rho^2)}]}{2\pi\sigma^2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} exp[\frac{-1}{2\sigma^2(1-\rho^2)}(y^2-2\rho xy+\rho x^2-\rho^2x^2)]dy, \\
&= \frac{exp[\frac{-x^2+\rho^2x^2}{2\sigma^2(1-\rho^2)}]}{2\pi\sigma^2\sqrt{\sigma\rho^2}} \int_{-\infty}^{\infty} e^{\frac{-(y-\rho x)}{2\sigma^2(1-\rho^2)}} dy, \\
&= \frac{exp[\frac{-(1-\rho^2)x^2}{2\sigma^2(1-\rho^2)}]}{\sqrt{2\pi}\sigma} \frac{\sqrt{1-\rho^{2^x}}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{-(y-\rho x)}{2\sigma^2(1-\rho^2)}} dy.
\end{aligned}$$

Because

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma}} dx = 1.$$

So if $\rho = 0$,

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}.$$

Similarly,

$$f_Y(y) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-y^2}{2\sigma^2}}.$$

We can have

$$f_{X,Y}(x,y) \quad = \quad f_X(x) f_Y(y).$$

So X *and* Y *are independent. If* $\rho \neq 0$, X *and* Y *are not independent.*