Rank-Metric Codes with Local Recoverability

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Abstract— We construct *rank-metric codes* with locality constraints under the rank-metric. Our motivation stems from designing codes for efficient data recovery from correlated and/or mixed (i.e., complete and partial) failures in distributed storage systems. Specifically, the proposed local rank-metric codes can recover locally from *crisscross failures*, which affect a limited number of rows and/or columns of the storage system. First, we prove a Singleton-like upper bound on the minimum rank-distance of linear codes with *rank-locality* constraints. Second, we construct a family of locally recoverable rank-metric codes that achieve this bound for a broad range of parameters. The proposed construction builds upon Tamo and Barg's method for constructing locally repairable codes with optimal minimum Hamming distance.

I. INTRODUCTION

Distributed storage systems have been traditionally replicating data over multiple nodes to guarantee reliability against failures and protect the data from being lost [1], [2]. However, the enormous growth of data being stored or computed online has motivated practical systems to employ erasure codes for handling failures (*e.g.*, [3], [4]). This has galvanized a significant amount of work in the past few years on novel erasure codes that efficiently handle node failures in distributed storage systems. Two main families of codes have received primary research attention: (a) *regenerating codes* – that minimize repair bandwidth, *i.e.*, the amount of data downloaded while repairing a failed node (see, *e.g.*, [5], [6], [7]); and (b) *locally repairable codes* – that minimize locality, *i.e.*, the number of nodes participating in the repair process (see, *e.g.*, [8], [9], [10], [11], [12]). Almost all the work in the literature on these families has considered block codes under the Hamming metric.

In this work, we focus our attention to codes with locality constraints in the rank-metric. Codewords of a rank-metric code are $m \times n$ matrices, where the rank-distance between two matrices is the rank of their difference [13], [14], [15]. Maximum rank-distance (MRD) codes are analogues of the maximum distance separable (MDS) codes in the Hamming metric. We are interested in rank-metric codes with locality constraints. To quantify the requirement of locality under the rank-metric, we introduce the notion of *rank-locality*. We say that the i-th column of an $m \times n$ array code has (r, δ) *rank-locality* if there exists a set of $r + \delta - 1$ columns containing

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i, which form an $[m \times (r + \delta - 1), r, \delta]$-MRD code. We say that an $m \times n$ array code has (r, δ) rank-locality if every column has (r, δ) rank-locality.

Our motivation of considering rank-locality is to design codes that can locally recover from *rank errors and erasures*. Rank-errors are the error patterns such that the rank of the error matrix is limited. For instance, consider an error pattern corrupting a 4×4 bit array shown in Fig. 1. Though this pattern corrupts half the bits, its rank over the binary field is only one. Note that it is not possible to correct such an error

$$
E \;=\; \begin{array}{|c|c|c|c|}
\hline
1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Fig. 1. A rank-error pattern of rank one.

pattern using a code equipped with the Hamming metric. On the other hand, rank-metric codes are well known for their ability to effectively correct rank-errors [15], [16].

Rank-erasures are the failure patterns that affect a limited number of rows and/or columns. Such patterns are also referred to as *crisscross patterns* [15], [16]. Supposing that the data is stored as an array, a crisscross failure pattern affects a few number of rows and/or columns (see Fig. 2 for some examples of crisscross erasures). Our goal is to investigate codes that can *locally* recover from crisscross erasures (and rank-errors). We note that crisscross errors (with no locality) have been studied previously in the literature [15], [16], motivated by applications in memory chip arrays and multi-track magnetic tapes. Our renewed interest in these types of failures stems from the fact that they form a subclass of *correlated and mixed failures*.

Recent research has shown that many distributed storage systems suffer from a large number of *correlated and mixed* failures [17], [18], [19], [20]. For instance, correlated failure of several nodes can occur due to, say, simultaneous upgrade of a group of servers, or a failure of a rack switch or a power supply shared by several nodes [17], [18]. Moreover, in distributed storage systems composed of solid state drives (SSDs), it is not uncommon to have a failed SSD along with a few corrupted *blocks* in the remaining SSDs, referred to as mixed failures [20], [21]. Therefore, recent research on coding for distributed storage has also started focusing on correlated and/or mixed failure models, see *e.g.*, [22], [23], [24], [25], [26], [21], [27], [28].

In general, our goal is to design and analyze codes that can *locally recover* the crisscross patterns that affect a limited number of rows and columns by accessing a small number of nodes. We show that a code with (r, δ) rank-locality can
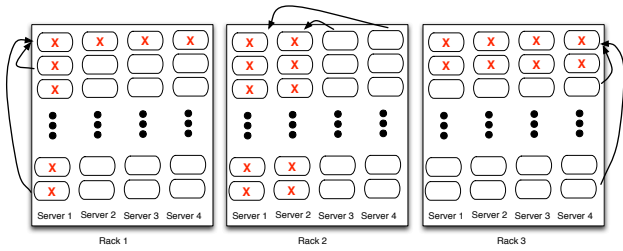
Fig. 2. Our motivation is to study codes for distributed storage systems that can locally recover from correlated and/or mixed failures, with particular focus on their subclass called crisscross failures. A crisscross failure pattern affects a limited number of rows and/or columns. For example, a few instances of crisscross failures affecting two rows and/or columns are depicted in the figure. We study rank-metric codes with local recoverability property as follow. Any crisscross failure pattern that affects at most δ rows and/or columns of a rack can be locally recovered by only accessing the nodes in the same rack.

locally repair any crisscross erasure pattern that affects less than δ rows and columns by accessing only r columns. We begin with a toy example to motivate the coding theoretic problem that we seek to solve.

Example 1: Consider a datacenter, such as the one depicted in Fig. 2, consisting of multiple racks, each of which containing a number of servers. Each server is composed of a number of storage nodes which can either be solid state drives (SSDs) or hard disk drives (HDDs).[1] Given two positive integers δ and d such that $\delta < d$, our goal is to encode the data in such a way that

1) any crisscross failure affecting at most δ rows and/or columns of nodes in a rack should be *'locally'* recoverable by only accessing the nodes on the corresponding rack, and

2) any crisscross failure that affects at most d rows and/or columns of nodes in the datacenter should be recoverable (potentially by accessing all the remaining data).

Note that the failure patterns of the first kind can occur in several cases. For example, all the nodes on a server would fail if, say, the network switch connecting the server to the system fails. The entire row of nodes would be temporarily unavailable if these nodes are simultaneously scheduled for an upgrade. A few locally recoverable crisscross patterns are shown in Fig. 2 (considering $\delta = 2$). Note that locally recoverable erasures in different racks can be simultaneously repaired.

A. Our Contributions

We begin with establishing a tight upper bound on the minimum rank-distance of codes with (r, δ) rank-locality. Then, we construct a family of *optimal* codes which achieve this upper bound. Our approach is inspired by [12], which generalizes Reed-Solomon code construction to obtain codes with locality. We generalize the Gabidulin code construction [14] to design codes with rank-locality. In particular, we obtain codes as evaluations of specially constructed

linearized polynomials over an extension field, and our codes reduce to Gabidulin codes if the locality parameter r equals the code dimension. Finally, we characterize various erasure and error patterns that the proposed codes with rank-locality can efficiently correct.

B. Related Work

Codes with Locality: Consider a block code of length n that encodes k information symbols. A symbol i is said to have *locality* r if it can be recovered by accessing r other symbols in the code. Note that r is the minimum possible size of a *recovering set* for the i-th symbol. We say that a code has locality r if each of its n symbols has locality at most r.

Codes with small locality were introduced in [8], [31] (see also [10]). The study of the locality property was galvanized with the pioneering work of Gopalan *et al.* [9]. One of their key contributions was to establish a trade-off between the minimum distance of a scalar linear code and its locality analogous to the classical Singleton bound.

The distance bound was generalized in several ways and a number of optimal code constructions have been proposed, see *e.g.*, [32], [12], [33], [34]. In particular, it is worth noting the following two references. In [32], the authors construct optimal LRCs using rank-metric codes as outer codes, and in [12] the authors generalize Reed-Solomon code construction to design LRCs with small alphabet size.[2]

Rank-Metric Codes: Rank-metric codes were introduced by Delsarte [13] and were largely developed by Gabidulin [14] (see also [15]). In addition, Gabidulin [14] presented a construction for a class of MRD codes. Roth [15] introduced the notion of crisscross error pattern, and showed that MRD codes are powerful in correcting such error patterns. In [16], the authors presented a family of MDS array codes for correcting crisscross errors.

Codes for Mixed Failures: Several families of codes have recently been proposed to encounter mixed failures. The two main families are: sector-disk (SD) codes and partial-MDS (PMDS) codes (see [21], [27], [28]). These codes consider the set up when data is stored in an $m \times n$ array, where a column of an array can be considered as an SSD. Each row of the array contains up to k data symbols and $h = n - k$ parity symbols, which together form a maximum distance separable (MDS) code. Furthermore, there are s global parity symbols in the first k columns. SD codes can tolerate erasure of any h drives, plus erasure of any additional s sectors in the array. PMDS codes can tolerate a broader class of erasures: any h sector erasures per row, plus any additional s sector erasures.

Sector-Disk (SD) codes for correcting mixed failures, *i.e.*, disk failures and sector failures, were introduced in [27]. Partial-MDS (PMDS) codes for correcting mixed failures were introduced in [21].

Codes for Correlated Failures: Very recently, Gopalan *et al.* [26] presented a class of *maximally recoverable* (MR) codes for *grid-like topologies*. For an $m \times n$ array, the grid-like topology essentially specifies the number of *local* parity

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[1]Many practical storage systems such as Facebook's 'F4' storage system [4] and all-flash storage arrays such as [29], [30] have similar architecture.

[2]We present a detailed comparison of our proposed constructions with those of [12] and [32] in Sections III-C and III-D, respectively.

check equations in every row and every column, and the number of *global* parity check equations in the array. The maximal recoverability means that the code has the *strongest* erasure correction capability that is possible with a given grid-like topology. The notion of maximal recoverability was first proposed by [8] and was generalized by [24]. We note that the class of codes considered in [26] can be used to correct mixed failures as well.

II. PRELIMINARIES

Notation: We use the following notation. For an integer l, $[l] = \{1, 2, \ldots, l\}$. For a vector $\mathbf{x}$, $\mathsf{wt}(\mathbf{x})$ denotes its Hamming weight, *i.e.*, $\mathsf{wt}(\mathbf{x}) = |\{i : \mathbf{x}(i) \neq 0\}|$. For a matrix H, $\mathsf{rank}(H)$ denotes its rank.

Let $\mathcal{C}$ denote a linear (n, k) code over $\mathbb{F}_q$ with block-length n, dimension k, and minimum distance $d_{min}(\mathcal{C})$. For instance, under Hamming metric, we have $d_{min}(\mathcal{C}) = \min_{\mathbf{c}_i, \mathbf{c}_j \in \mathcal{C}, \, \mathbf{c}_i \neq \mathbf{c}_j} \mathsf{wt}(\mathbf{c}_i - \mathbf{c}_j)$. Given a length-$n$ block code $\mathcal{C}$ and a set $\mathcal{S} \subset [n]$, let $\mathcal{C}|_{\mathcal{S}}$ denote the restriction of $\mathcal{C}$ on the coordinates in $\mathcal{S}$. Essentially, $\mathcal{C}|_{\mathcal{S}}$ is the code obtained by puncturing $\mathcal{C}$ on $[n] \setminus \mathcal{S}$.

Recall that, for Hamming metric, the well known Singleton bound gives an upper bound on the minimum distance of an (n, k) code $\mathcal{C}$ as $d_{min}(\mathcal{C}) \leq n - k + 1$. Codes which meet the Singleton bound are called maximum distance separable (MDS) codes (see, *e.g.*, [35]).

A. Rank-Metric Codes

Let $\mathbb{F}_q^{m \times n}$ be the set of all $m \times n$ matrices over $\mathbb{F}_q$. The *rank-distance* is a distance measure between elements A and B of $\mathbb{F}_q^{m \times n}$, defined as $d_R(A, B) = \mathsf{rank}(A - B)$. It can be shown that the rank-distance is indeed a metric [14]. A rank-metric code is a non-empty subset of $\mathbb{F}_q^{m \times n}$ equipped with the rank-distance metric (see [13], [14], [15]). Rank-metric codes can be considered as array codes or matrix codes.

Typically, rank-metric codes are considered by leveraging the correspondence between $\mathbb{F}_q^{m \times 1}$ and the extension field $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$. In particular, by fixing a basis for $\mathbb{F}_{q^m}$ as an m-dimensional vector space over $\mathbb{F}_q$, any element of $\mathbb{F}_{q^m}$ can be represented as a length-m vector over $\mathbb{F}_q$. Similarly, any length-n vector over $\mathbb{F}_{q^m}$ can be represented as an $m \times n$ matrix over $\mathbb{F}_q$. The rank of a vector $\mathbf{a} \in \mathbb{F}_{q^m}^n$ is the rank of the corresponding $m \times n$ matrix A over $\mathbb{F}_q$. This rank does not depend on the choice of basis for $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. This correspondence allows us to view a rank-metric code in $\mathbb{F}_q^{m \times n}$ as a block code of length n over $\mathbb{F}_{q^m}$. More specifically, a rank metric code $\mathcal{C} \subseteq \mathbb{F}_q^{m \times n}$ is a block code of length n $\mathbb{F}_{q^m}$.

The Singleton bound for the rank-metric (see [14]) states that every rank-metric code with minimum rank-distance d must satisfy

$$|\mathcal{C}| \leq q^{\max\{n,m\}(\min\{n,m\} - d + 1)}.$$

Notice that the minimum rank-distance of a code $\mathcal{C}$ is given as

$$d_R(\mathcal{C}) = \min_{C_i, \, C_j \in \mathcal{C}, \, C_i \neq C_j} d_R(C_i, C_j).$$

Codes that achieve this bound are called maximum rank-distance (MRD) codes. Note that, for $m \geq n$, the Singleton

bound for rank metric coincides with the classical Singleton bound for the Hamming metric. Indeed, when $m \geq n$, every $[m \times n, k, d]$ MRD code over $\mathbb{F}_q$ is also an $[n, k, d]$ MDS code over $\mathbb{F}_{q^m}$, and hence can correct any $d - 1$ *column erasures*.

1) Gabidulin Codes: Gabidulin [14] presented a construction of a class of MRD codes for $m \geq n$. The construction is based on the evaluation of a special type of polynomials called *linearized polynomials*. For notational convenience, we write $x^{q^i} = x^{[i]}$.

Definition 1: [Linearized Polynomial] ([36]) A polynomial in $\mathbb{F}_{q^m}[x]$ of the following form

$$L(x) = \sum_{i=0}^{n} a_i x^{[i]} \tag{1}$$

is called a linearized polynomial, or a q-polynomial, over $\mathbb{F}_{q^m}$. Further, $\max\{i \in [n] \mid a_i \neq 0\}$ is said to be the q-degree of $L(x)$, denoted as $\mathsf{deg}_q(L(x))$.

Gabidulin Code Construction: An (n, k) Gabidulin code over the extension field $\mathbb{F}_{q^m}$ for $m \geq n$ is the set of evaluations of all q-polynomials of q-degree at most $k - 1$ over n elements of $\mathbb{F}_{q^m}$ that are linearly independent over $\mathbb{F}_q$.

In particular, let $P = \{p_1, \cdots, p_n\}$ be a set of n elements in $\mathbb{F}_{q^m}$ that are linearly independent over $\mathbb{F}_q$ ($m \geq n$). Let $G_{\mathbf{m}}(x) \in \mathbb{F}_{q^m}[x]$ denote the linearized polynomial of q-degree at most $k - 1$ with coefficients $\mathbf{m}$ as follows.

$$G_{\mathbf{m}}(x) = \sum_{j=0}^{k-1} \mathbf{m}_j x^{[j]}, \tag{2}$$

Then, the Gabidulin code is obtained by the following evaluation map

$$\begin{aligned} Enc : \mathbb{F}_{q^m}^k &\to \mathbb{F}_{q^m}^n \\ \mathbf{m} &\mapsto \{G_{\mathbf{m}}(\gamma), \gamma \in P\} \end{aligned} \tag{3}$$

Therefore, we have

$$\mathcal{C}_{Gab} = \left\{ (G_{\mathbf{m}}(\gamma), \gamma \in P) \mid \mathbf{m} \in \mathbb{F}_{q^m}^k \right\}. \tag{4}$$

Reed-Solomon Code Construction: It is worth mentioning the analogy between Reed-Solomon codes and Gabidulin codes. An (n, k) Reed-Solomon code over the finite field $\mathbb{F}_q$ for $q \geq n$ is the set of evaluations of all polynomials of degree at most $k - 1$ over n distinct elements of $\mathbb{F}_q$. More specifically, let $P = \{p_1, \cdots, p_n\}$ be a set of n distinct elements of $\mathbb{F}_q$ ($q \geq n$). Consider polynomials $g_{\mathbf{m}}(x) \in \mathbb{F}_q[x]$ of the following form

$$g_{\mathbf{m}}(x) = \sum_{j=0}^{k-1} \mathbf{m}_j x^j, \tag{5}$$

Then, the Reed-Solomon code is obtained by the following evaluation map

$$\begin{aligned} Enc : \mathbb{F}_q^k &\to \mathbb{F}_q^n \\ \mathbf{m} &\mapsto \{g_{\mathbf{m}}(\gamma), \gamma \in P\} \end{aligned} \tag{6}$$

Therefore, we have

$$\mathcal{C}_{RS} = \left\{ (g_{\mathbf{m}}(\gamma), \gamma \in P) \mid \mathbf{m} \in \mathbb{F}_q^k \right\}. \tag{7}$$

Remark 1: For the same information vector $\mathbf{m} = [m_0 \cdots m_{k-1}]$, the evaluation polynomials of the Gabidulin code and the Reed-Solomon code are q-associates of each other.

B. Codes with Locality

Locality of of a code captures the number of symbols participating in recovering a lost symbol. In particular, an (n, k) code is said to have locality r if every symbol is recoverable from a set of at most r symbols. For linear codes with locality, essentialy a *local parity check code* of length at most $r + 1$ is associated with every symbol. The notion of locality can be generalized to accommodate local codes of larger distance as follows (see [37]).

Definition 2: [Locality] An (n, k) code $\mathcal{C}$ is said to have (r, δ) locality, if for each symbol $\mathbf{c}_i$, $i \in [n]$, of a codeword $\mathbf{c} = [\mathbf{c}_1 \, \mathbf{c}_2 \, \cdots \mathbf{c}_n] \in \mathcal{C}$, there exists a set of indices $\Gamma(i)$ such that

1) $i \in \Gamma(i)$,
2) $|\Gamma(i)| \leq r + \delta - 1$, and
3) $d_{min}(\mathcal{C}|_{\Gamma(i)}) \geq \delta$.

The code $d_{min}(\mathcal{C}|_{\Gamma(i)})$ is said to be the local code associated with the i-th coordinate of $\mathcal{C}$.

Properties 2 and 3 imply that for any codeword in $\mathcal{C}$, the values in $\Gamma(i)$ are uniquely determined by any r of those values. Under Hamming metric, the (r, δ) locality allows one to repair any $\delta - 1$ erasures in $\mathcal{C}|_{\Gamma(i)}$, $\forall i \in [n]$, *locally* by accessing at most r other symbols. When $\delta = 2$, the above definition reduces to the classical definition of locality proposed by Gopalan *et al.* [9], wherein any one erasure can be repaired by accessing at most r symbols. The Singleton bound can be generalized to accommodate locality constraints. In particular, the minimum Hamming distance of an (n, k) code $\mathcal{C}$ with (r, δ) locality is upper bounded as follows (see [38, Theorem 21], also [37, Theorem 2] for linear codes):

$$d_{min}(\mathcal{C}) \leq n - k + 1 - \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right)(\delta - 1). \quad (8)$$

Most of the existing work on locally recoverable codes has been focused on locality with respect to Hamming metric. We are interested in locality with respect to rank-metric.

III. CODES WITH RANK-LOCALITY

Recall from Definition 2 that, for a code $\mathcal{C}$ with (r, δ) locality, the *local code* $\mathcal{C}|_{\Gamma(i)}$ associated with the i-th symbol, $i \in [n]$, has minimum distance at least δ. We are interested in rank-metric codes such that the local code associated with every column should be a rank-metric code with minimum rank-distance guarantee. This motivates us to generalize the concept of locality to that of *rank-locality* as follows.

Definition 3 (Rank-Locality): An $(m \times n, k)$ rank-metric code $\mathcal{C}$ is said to have (r, δ) rank-locality if for each column $i \in [n]$ of the codeword matrix, there exists a set of columns $\Gamma(i) \subset [n]$ such that

1) $i \in \Gamma(i)$,
2) $|\Gamma(i)| \leq r + \delta - 1$, and

3) $d_R(\mathcal{C}|_{\Gamma(i)}) \geq \delta$,

where $\mathcal{C}|_{\Gamma(i)}$ is the restriction of $\mathcal{C}$ on the columns of $\Gamma(i)$. The code $\mathcal{C}|_{\Gamma(i)}$ is said to be the local code associated with the i-th column. An $(m \times n, k)$ rank-metric code with (r, δ) locality is denoted as an $(m \times n, k, r, \delta)$ rank-metric code.

As we will see in Section IV, the (r, δ)-rank-locality allows us to repair any crisscross erasure pattern of weight $\delta - 1$ in $\mathcal{C}|_{\Gamma(i)}$, $\forall i \in [n]$, *locally* by accessing the symbols of $\mathcal{C}|_{\Gamma(i)}$. Further, we can correct any crisscross erasure pattern of weight $d_R(\mathcal{C}) - 1$ in $\mathcal{C}$ by accessing unerased symbols of $\mathcal{C}$.

Remark 2: In the remainder of the paper, we assume that the columns of an $(m \times n, k, r, \delta)$ rank-metric code $\mathcal{C}$ can be partitioned into $\mu := n/(r + \delta - 1)$ disjoint sets $C_1, \ldots, C_\mu$ each of size $r + \delta - 1$ such that, for all $i \in C_j$, $\Gamma(i) = C_j$. In other words, we assume that the local codes associated with the columns have disjoint coordinates.

A. Upper Bound on Rank Distance

It is easy to find the Singleton-like upper bound on the minimum rank-distance for codes with rank-locality using the results in the Hamming metric.

Theorem 1: For a rank-metric code $\mathcal{C} \subseteq \mathbb{F}_q^{m \times n}$ of cardinality q^{mk} with (r, δ) rank-locality, we have

$$d_R(\mathcal{C}) \leq n - k + 1 - \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right)(\delta - 1). \quad (9)$$

Proof: Note that by fixing a basis for $\mathbb{F}_{q^m}$ as a vector space over $\mathbb{F}_q$, we can obtain a bijection $\phi : \mathbb{F}_{q^m} \to \mathbb{F}_q^{m \times 1}$. This can be extended to a bijection $\phi : \mathbb{F}_{q^m}^n \to \mathbb{F}_q^{m \times n}$. Then, for any vector $\mathbf{c} \in \mathbb{F}_{q^m}^n$, there is a corresponding matrix $C \in \mathbb{F}_q^{m \times n}$ such that $C = \phi(\mathbf{c})$. For any such vector-matrix pair, we have

$$\mathsf{rank}(C) \leq \mathsf{wt}(\mathbf{c}). \quad (10)$$

An $(m \times n, k, d)$ rank-metric code $\mathcal{C}$ over $\mathbb{F}_q$ can be considered as a block code of length n over $\mathbb{F}_{q^m}$, denoted as $\mathcal{C}'$. From (10), it follows that $d_R(\mathcal{C}) \leq d_{min}(\mathcal{C}')$. Moreover, it follows that, if $\mathcal{C}$ has (r, δ) rank-locality, then the corresponding code $\mathcal{C}'$ possesses (r, δ) locality in the Hamming metric. Therefore, an upper bound on the minimum Hamming distance of an (n, k, d')-LRC $\mathcal{C}'$ with (r, δ) locality is also an upper bound on the rank-distance of an $(m \times n, k, d)$ rank-metric code with (r, δ) rank-locality. Hence, (9) follows from (8). ∎

B. Code Construction

We build upon the construction methodology of Tamo and Barg [12] to construct codes with rank-locality that are optimal with respect to the rank-distance bound in (9). In particular, the codes are constructed as the evaluations of specially designed linearized polynomials on a specifically chosen set of points of $\mathbb{F}_{q^m}$. The detailed construction is as follows.

Construction 1: [$(m \times n, k, r, \delta)$ rank-metric code.] Let n, k, r, δ be positive integers such that $r \mid k$, $(r + \delta - 1) \mid n$, and $n \mid m$. Define $\mu := n/(r + \delta - 1)$. Fix $q \geq 2$ to be a power of a prime. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_{r+\delta-1}\}$ be a basis of $\mathbb{F}_{q^{r+\delta-1}}$ as a vector space over $\mathbb{F}_q$, and $\mathcal{B} = \{\beta_1, \ldots, \beta_\mu\}$ be a basis of $\mathbb{F}_{q^n}$ as a vector space over $\mathbb{F}_{q^{r+\delta-1}}$. Define

the set of n evaluation points $P \subset \mathbb{F}_{q^m}$, with the partition $\mathcal{P} = P_1 \cup \cdots \cup P_\mu$, where $P_j = \{\alpha_i \beta_j, 1 \leq i \leq r + \delta - 1\}$ for $1 \leq j \leq \mu$. To encode the message $\mathbf{m} \in \mathbb{F}_{q^m}^k$, denoted as $\mathbf{m} = \{m_{ij} : i = 0, \ldots, r-1; j = 0, \ldots, k/r - 1\}$, define the encoding polynomial

$$G_{\mathbf{m}}(x) = \sum_{i=0}^{r-1} \sum_{j=0}^{\frac{k}{r}-1} \mathbf{m}_{ij} x^{[(r+\delta-1)j+i]}. \tag{11}$$

The codeword for $\mathbf{m}$ is obtained as the vector of the evaluations of $G_{\mathbf{m}}(x)$ at all the points of P. In other words, the linear code $\mathcal{C}_{Loc}$ is constructed as the following evaluation map:

$$\begin{aligned} Enc : \mathbb{F}_{q^m}^k &\to \mathbb{F}_{q^m}^n \\ \mathbf{m} &\mapsto \{G_{\mathbf{m}}(\gamma), \gamma \in P\}. \end{aligned} \tag{12}$$

Therefore, we have

$$\mathcal{C}_{Loc} = \left\{ (G_{\mathbf{m}}(\gamma), \gamma \in P) \mid \mathbf{m} \in \mathbb{F}_{q^m}^k \right\}. \tag{13}$$

The $(m \times n, k)$ rank-metric code is obtained by considering the matrix representation of every codeword obtained as above by fixing a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. We denote the following μ codes as the local codes.

$$\mathcal{C}_j = \left\{ (G_{\mathbf{m}}(\gamma), \gamma \in P_j) \mid \mathbf{m} \in \mathbb{F}_{q^m}^k \right\}, \quad 1 \leq j \leq \mu. \tag{14}$$

Remark 3: [Field Size] It is worth mentioning that, even for constructing Gabidulin codes of length n over $\mathbb{F}_{q^m}$, it is required that $m \geq n$ [14]. Note that, it is sufficient to choose $m = n$ and $q = 2$ in our construction. In other words, the field size of 2^n is sufficient for the proposed code construction.

In the following, we show that Construction 1 gives codes with rank-locality, which are optimal with respect to the rank-distance bound in Theorem 1.

Theorem 2: Construction 1 gives a linear $[m \times n, k, d]$ rank-metric code $\mathcal{C}_{Loc}$ with (r, δ) rank-locality such that the minimum rank-distance d is equal to the upper bound given in (9).

Proof: The proof makes use of two key lemmas; Lemma 1 is used to prove the the rank-distance optimality and Lemma 2 is used to prove the rank-locality for the proposed construction.

We begin with showing the rank-distance optimality of $\mathcal{C}_{Loc}$. The first step is to prove the linear independence of the evaluation points P as stated in the following lemma.

Lemma 1: The n evaluation points given in Construction 1, $P = \{\alpha_i \beta_j, 1 \leq i \leq r + \delta - 1, 1 \leq j \leq \mu\}$, are linearly independent over $\mathbb{F}_q$.

Proof: Suppose, for contradiction, that the evaluation points are linearly dependent over $\mathbb{F}_q$. Then, we have $\sum_{j=1}^{\mu} \sum_{i=1}^{r+\delta-1} \omega_{ij} \alpha_i \beta_j = 0$ with coefficients $\omega_{ij} \in \mathbb{F}_q$ such that not all ω_{ij}'s are zero. We can write the linear dependence condition as $\sum_{j=1}^{\mu} \left(\sum_{i=1}^{r+\delta-1} \omega_{ij} \alpha_i \right) \beta_j = 0$. Now, from the linear independence of the β_j's over $\mathbb{F}_{q^{r+\delta-1}}$, we have $\sum_{i=1}^{r+\delta-1} \omega_{ij} \alpha_i = 0$ for each $1 \leq j \leq \mu$. However, as the α_i's are linearly independent over $\mathbb{F}_q$, we have every $\omega_{ij} = 0$. This is a contradiction. ∎

Lemma 1 essentially asserts that $\mathcal{C}_{Loc}$ is obtained as the evaluations of $G_{\mathbf{m}}(x)$ on n points of $\mathbb{F}_{q^m}$ that are linearly independent over $\mathbb{F}_q$. Combining this with the structure of $G_{\mathbf{m}}(x)$ (see (11)), $\mathcal{C}_{Loc}$ can be considered as a subcode of an $(n, k + (k/r - 1)(\delta - 1))$ Gabidulin code (cf. (2)). Hence, $d_R(\mathcal{C}_{Loc}) \geq n - k + 1 - (k/r - 1)(\delta - 1)$, which shows $d_R(\mathcal{C}_{Loc})$ attains the upper bound (9) in Theorem 1, and thus that the proposed construction is optimal with respect to rank-distance.

Second, we show that $\mathcal{C}_{Loc}$ has (r, δ) rank-locality. Define $H(x) = x^{q^{r+\delta-1}-1} = x^{[r+\delta-1]-1}$. We note that (11) can be written in the following form using $H(x)$.

$$G_{\mathbf{m}}(x) = \sum_{i=0}^{r-1} G_i(x) x^{[i]}, \tag{15}$$

where

$$G_i(x) = \mathbf{m}_{i0} + \sum_{j=1}^{\frac{k}{r}-1} \mathbf{m}_{ij} [H(x)]^{\sum_{l=0}^{j-1} q^{(r+\delta-1)l+i}}. \tag{16}$$

To see this, observe that

$$\begin{aligned} [H(x)]^{\sum_{l=0}^{j-1} q^{(r+\delta-1)l+i}} &= \left[x^{q^{r+\delta-1}-1} \right]^{\sum_{l=0}^{j-1} q^{(r+\delta-1)l+i}} \\ &= x^{\sum_{l=0}^{j-1} q^{(r+\delta-1)(l+1)+i} - \sum_{l=0}^{j-1} q^{(r+\delta-1)l+i}} \\ &= x^{q^{(r+\delta-1)j+i} - q^i}. \end{aligned} \tag{17}$$

Using this in (16), we get

$$G_i(x) = \mathbf{m}_{i0} + \sum_{j=1}^{\frac{k}{r}-1} \mathbf{m}_{ij} x^{[(r+\delta-1)j+i]-[i]}. \tag{18}$$

Substituting (18) into (15) gives us (11).

Now, to prove the rank-locality, we want to show that $d_R(\mathcal{C}_j) \geq \delta$ for every local code $1 \leq j \leq \mu$. Towards this, let $\gamma \in P_j$ and define the *repair polynomial* as

$$R_j(x) = \sum_{i=0}^{r-1} G_i(\gamma) x^{[i]}. \tag{19}$$

We show that $\mathcal{C}_j$ can be considered as obtained by evaluating $R_j(x)$ on the points of P_j. For this, we first prove that $H(x)$ is constant on all points of P_j for each $1 \leq j \leq \mu$.

Lemma 2: Consider the partition of the set of evaluation points given in Construction 1 as $\mathcal{P} = P_1 \cup \cdots \cup P_\mu$, where $P_j = \{\alpha_i \beta_j, 1 \leq i \leq r + \delta - 1\}$. Then, $H(x)$ is constant on all evaluation points of any set P_j, $1 \leq j \leq \mu$.

Proof: Note that $H(\beta_j \alpha_i) = (\beta_j \alpha_i)^{[r+\delta-1]-1} = \beta_j^{[r+\delta-1]-1} \alpha_i^{[r+\delta-1]-1} = \beta_j^{[r+\delta-1]-1}$, where the last equality follows from $\alpha_i \in \mathbb{F}_{q^{r+\delta-1}} \setminus \{0\}$. Thus, $H(\omega) = \beta_j^{[r+\delta-1]-1}$, for all $\omega \in P_j$, $1 \leq j \leq \mu$. ∎

Note that, since $G_i(x)$ is a linear combination of powers of $H(x)$, it is also constant on the set P_j. In other words, we have

$$G_i(\gamma) = G_i(\lambda), \quad \forall \gamma, \lambda \in P_j, \tag{20}$$

for every $0 \leq i \leq r - 1$.

Moreover, when evaluating $R_j(x)$ in $\lambda \in P_j$, we get

$$R_j(\lambda) = \sum_{i=0}^{r-1} G_i(\gamma)\lambda^{[i]} = \sum_{i=0}^{r-1} G_i(\lambda)\lambda^{[i]} = G_{\mathbf{m}}(\lambda). \quad (21)$$

Hence, the evaluations of the encoding polynomial $G_{\mathbf{m}}(x)$ and the repair polynomial $R_j(x)$ on points in P_j are identical. In other words, we can consider that $\mathcal{C}_j$ is obtained by evaluating $R_j(x)$ on points of P_j. Now, since points of P_j are linearly independent over $\mathbb{F}_q$, and $R_j(x)$ is a linearized polynomial of q-degree $r - 1$, $\mathcal{C}_j$ can be considered as a $(r + \delta - 1, r)$ Gabidulin code (cf. (2)). Thus, $\mathcal{C}_j$ is an MRD code, and we have $d_R(\mathcal{C}_j) = \delta$, which proves the rank-locality of the proposed construction. This concludes the proof of Theorem 2. $\blacksquare$

Next, we present an example of an $(9 \times 9, 4)$ rank-metric code with $(2, 2)$ rank-locality. We note that the code presented in this example satisfies the correctability constraints specified in the motivating example (Example 1) in the introduction section.

Example 2: Let $n = 9, k = 4, r = 2, \delta = 2$. Set $q = 2$ and $m = n$. Let ω be the primitive element of $\mathbb{F}_{2^9}$ with respect to the primitive polynomial $p(x) = x^9 + x^4 + 1$. Note that ω^{73} generates $\mathbb{F}_{2^3}$, as $\left(\omega^{73}\right)^7 = 1$. Consider $\mathcal{A} = \{1, \omega^{73}, \omega^{146}\}$ as a basis for $\mathbb{F}_{2^3}$ over $\mathbb{F}_2$. We view $\mathbb{F}_{2^9}$ as an extension field over $\mathbb{F}_{2^3}$ considering the irreducible polynomial $p(x) = x^3 + x + \omega^{73}$. It is easy to verify that ω^{309} is a root of $p(x)$, and thus, $\mathcal{B} = \{1, \omega^{309}, \omega^{107}\}$ forms a basis of $\mathbb{F}_{2^9}$ over $\mathbb{F}_{2^3}$. Then, the evaluation points P and their partition $\mathcal{P}$ is as follows.

$$\mathcal{P} = \{\{1, \omega^{73}, \omega^{146}\}, \{\omega^{309}, \omega^{382}, \omega^{455}\}, \{\omega^{107}, \omega^{180}, \omega^{253}\}\}.$$

Let $\mathbf{m} = (\mathbf{m}_{00}, \mathbf{m}_{01}, \mathbf{m}_{10}, \mathbf{m}_{11}) \in \mathbb{F}_{2^9}^4$ be the information vector. Define the encoding polynomial (as in (11)) as follows.

$$G_{\mathbf{m}}(x) = \mathbf{m}_{00}x^{[0]} + \mathbf{m}_{01}x^{[3]} + \mathbf{m}_{10}x^{[1]} + \mathbf{m}_{11}x^{[4]}.$$

The codeword $\mathbf{c}$ for the information vector $\mathbf{m}$ is obtained as the evaluation of the polynomial $G_{\mathbf{m}}(x)$ at all the points of P. The code $\mathcal{C}$ is the set of codewords corresponding to all $\mathbf{m} \in \mathbb{F}_{2^9}^4$.

From Lemma 1, the evaluation points are linearly independent over $\mathbb{F}_2$, and thus, $\mathcal{C}$ can be considered as a subcode of a $(9, 5)$ Gabidulin code (cf. (2)). Thus, $d_R(\mathcal{C}) = 5$, which is optimal with respect to (9).

Now, consider the local codes $\mathcal{C}_j$, $1 \le j \le 3$. It is easy to verify that $\mathcal{C}_j$ can be obtained by evaluating the *repair polynomial* $R_j(x)$ on P_j given as follows (see (19)).

$$R_1(x) = (\mathbf{m}_{00} + \mathbf{m}_{01})x^{[0]} + (\mathbf{m}_{10} + \mathbf{m}_{11})x^{[1]},$$
$$R_2(x) = (\mathbf{m}_{00} + \omega^{119}\mathbf{m}_{01})x^{[0]} + (\mathbf{m}_{10} + \omega^{238}\mathbf{m}_{11})x^{[1]},$$
$$R_3(x) = (\mathbf{m}_{00} + \omega^{238}\mathbf{m}_{01})x^{[0]} + (\mathbf{m}_{10} + \omega^{476}\mathbf{m}_{11})x^{[1]}.$$

For instance, let the message vector be $\mathbf{m} = (\omega, \omega^2, \omega^4, \omega^8)$. Then, the codeword is

$$\mathbf{c} = (\omega^{440}, \omega^{307}, \omega^{81}, \omega^{465}, \omega^{11}, \omega^{174}, \omega^{236}, \omega^{132}, \omega^{399}).$$

One can easily check that evaluating $R_1(x)$ on P_1 gives $\mathbf{c}_1 = (\omega^{440}, \omega^{307}, \omega^{81})$, evaluating $R_2(x)$ on P_2 gives $\mathbf{c}_2 = (\omega^{465}, \omega^{11}, \omega^{174})$, and evaluating $R_3(x)$ on P_3 gives $\mathbf{c}_3 = (\omega^{236}, \omega^{132}, \omega^{399})$.

This implies that the local code $\mathcal{C}_j$, $1 \le j \le 3$, can be considered as obtained by evaluating a linearized polynomial of the form $R_j(x) = \mathbf{m}'_0 x^{[0]} + \mathbf{m}'_1 x^{[1]}$ on three points that are linearly independent over $\mathbb{F}_2$. Hence, $\mathcal{C}_j$ is a Gabidulin code of length 3 and dimension 2, which gives $d_R(\mathcal{C}|_{P_i}) = 2$. This shows that $\mathcal{C}$ has $(2, 2)$ rank-locality.

C. Comparison with Tamo and Barg [12]

The key idea in [12] is to construct codes with locality as evaluations of a specially designed polynomial over a specifically chosen set of elements of the underlying finite field. To point out the similarities and differences, we briefly review Construction 8 from [12]. We assume that $r \mid k$, and $r + \delta - 1 \mid n$.

Construction 8 from [12]: Let $\mathcal{P} = \{P_1, \dots, P_\mu\}$, $\mu = n/(r + \delta - 1)$, be a partition of the set $P \subset \mathbb{F}_q$, $|P| = n$, such that $|P_i| = r + \delta - 1$, $1 \le i \le \mu$. Let $h \in \mathbb{F}_q[x]$ be a polynomial of degree $r + \delta - 1$, called the *good polynomial*, that is constant on each of the sets P_i. For an information vector $\mathbf{m} \in \mathbb{F}_q^k$, define the encoding polynomial

$$g_{\mathbf{m}}(x) = \sum_{i=0}^{r-1} \left(\sum_{j=0}^{\frac{k}{r}-1} \mathbf{m}_{ij} h(x)^j \right) x^i.$$

The code $\mathcal{C}$ is defined as the set of n-dimensional vectors

$$\mathcal{C} = \left\{ (g_{\mathbf{m}}(\gamma), \gamma \in P) \mid \mathbf{m} \in \mathbb{F}_q^k \right\}.$$

The authors show that $h(x) = x^{r+\delta-1}$ can be used as a *good polynomial*, when the evaluation points are the cosets of a multiplicative subgroup of $\mathbb{F}_q^*$ of order $r + \delta - 1$. In this case, we can write $g_{\mathbf{m}}(x)$ as

$$g_{\mathbf{m}}(x) = \sum_{i=0}^{r-1} \sum_{j=0}^{\frac{k}{r}-1} \mathbf{m}_{ij} x^{(r+\delta-1)j+i}. \quad (22)$$

Therefore, $\mathcal{C}$ can be considered as a subcode of an $(n, k + (^k/_r - 1)(\delta - 1))$ Reed-Solomon code. In addition, local codes $\mathcal{C}_j = \left\{ (g_{\mathbf{m}}(\gamma), \gamma \in P_j) \mid \mathbf{m} \in \mathbb{F}_q^k \right\}$, $1 \le j \le \mu$, can be considered as $(r + \delta - 1, r)$ Reed-Solomon codes.

In our case, the code $\mathcal{C}_{Loc}$ obtained from Construction 1 can be considered as a subcode of a $(n, k + (^k/_r - 1)(\delta - 1))$ Gabidulin code. Further, the local codes $\mathcal{C}_j$, $1 \le j \le \mu$, can be considered as $(r + \delta - 1, r)$ Gabidulin codes. In fact, as one can see from the proof of Theorem 2, we implicitly use $H(x) = x^{[r+\delta-1]-1}$ as the good polynomial, which evaluates as a constant on all points of P_j for $1 \le j \le \mu$ given in Construction 1. It is worth mentioning that (22) and (11) turn out to be q-associates of each other.

D. Comparison with Silberstein et al. [32]

In [32] (see also [38]), the authors have presented a construction of LRC codes based on rank-metric codes. The idea is to precode the information vector with an $(r\mu, k)$

??	??	??	??	??	??	?	?	?
??	??	??	??	$c_{2,5}$	$c_{2,6}$	$c_{2,7}$	$c_{2,8}$	$c_{2,9}$
??	??	??	??	$c_{3,5}$	$c_{3,6}$	$c_{3,7}$	$c_{3,8}$	$c_{3,9}$
$c_{4,1}$	$c_{4,2}$	$c_{4,3}$	??	$c_{4,5}$	$c_{4,6}$	$c_{4,7}$	$c_{4,8}$	$c_{4,9}$
$c_{5,1}$	$c_{5,2}$	$c_{5,3}$	??	$c_{5,5}$	$c_{5,6}$	$c_{5,7}$	$c_{5,8}$	$c_{5,9}$
$c_{6,1}$	$c_{6,2}$	$c_{6,3}$	??	$c_{6,5}$	$c_{6,6}$	$c_{6,7}$	$c_{6,8}$	$c_{6,9}$
$c_{7,1}$	$c_{7,2}$	$c_{7,3}$	??	$c_{7,5}$	$c_{7,6}$	$c_{7,7}$	$c_{7,8}$	$c_{7,9}$
$c_{8,1}$	$c_{8,2}$	$c_{8,3}$	??	$c_{8,5}$	$c_{8,6}$	$c_{8,7}$	$c_{8,8}$	$c_{8,9}$
$c_{9,1}$	$c_{9,2}$	$c_{9,3}$	??	$c_{9,5}$	$c_{9,6}$	$c_{9,7}$	$c_{9,8}$	$c_{9,9}$

Fig. 3. An example of a 9×9 bit array. When the erasure patterns affects a single row or column in a local array, it should be corrected locally. Further, any erasure pattern that is confined to at most four rows or columns (or both) should be globally correctable. In the example above, locally correctable erasures are denoted as '?', while globally correctable erasures are denoted as '??'.

Gabidulin code over $\mathbb{F}_{q^m}$. The symbols of the codeword are partitioned into μ sets $C_1, \ldots, C_\mu$ of size r each. For each set C_j, an $(r + \delta - 1, r)$ Reed-Solomon code over $\mathbb{F}_q$ is used to obtain $\delta - 1$ local parities, which together with the symbols of C_j forms the codeword of a local code $\mathcal{C}_j$. This ensures that each local code has minimum distance δ. However, it does not guarantee that the minimum rank-distance of a local code is at least δ.

In fact, for any $\mathbf{c} \in \mathcal{C}_j$, $1 \leq j \leq \mu$, we have $\mathsf{rank}(\mathbf{c}) \leq r$, as the local parities are obtained via linear combinations over $\mathbb{F}_q$. Clearly, when $\delta > r$, the construction cannot achieve rank-locality. Moreover, even if $\delta \leq r$, it is possible to have a codeword $\mathbf{c} \in \mathcal{C}_j$ such that $\mathsf{rank}(\mathcal{C}_i) < \delta$ for some local code $\mathcal{C}_j$. Therefore, in general, the construction of [32], that uses Gabidulin codes as outer codes, does not guarantee that the codes possess rank-locality.

On the other hand, our construction can be viewed as a method to design (n, k) linear codes over $\mathbb{F}_{q^m}$ with (r, δ) locality (under Hamming metric). For the construction in [32], the field size of q^n is sufficient for $q \geq r + \delta - 1$ when $\delta > 2$, while one can choose any $q \geq 2$ when $\delta = 2$. When our construction is used to obtain LRCs, it is sufficient to operate over the field of size 2^n.

IV. Erasure Correction Capability of Codes with Rank-Locality

Suppose the encoded data is stored on an $m \times n$ array C using an $(m \times n, k, d, r, \delta)$ rank-metric code $\mathcal{C}$ over $\mathbb{F}_{q^m}$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_\mu$ be the local codes of $\mathcal{C}$ and let $C_1, \ldots, C_\mu$ be the corresponding local arrays, where a local array C_i is of size $m \times (n/\mu)$. Our goal is to characterize the class of (possibly correlated) mixed erasure patterns corresponding to column and row failures of C that $\mathcal{C}$ can correct locally or globally. Towards this, we consider the notion of crisscross weight of an erasure pattern.

Let $E = [e_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ be an $m \times n$ binary matrix that specifies the location of the erased symbols of C. In particular, $e_{ij} = 1$ if the (i, j)-th entry of C is erased, otherwise $e_{ij} = 0$. For simplicity, we denote the erasure pattern by E itself. We denote by $E(C_j)$ the n/μ columns of E corresponding to the local array C_j, and we refer to $E(C_j)$ as the error pattern restricted to the local array C_j. We first consider the notion of a cover of E, which is used to define the crisscross weight of E (see [15], also [16]).

Definition 4: [Cover of E] A cover of an $m \times n$ matrix E is a pair (X, Y) of sets $X \subseteq [m]$, $Y \subseteq [n]$, such that $e_{ij} \neq 0 \implies ((i \in X) \text{ or } (j \in Y))$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. The size of the cover (X, Y) is defined as $|(X, Y)| = |X| + |Y|$.

We define the crisscross weight of an erasure pattern as the crisscross weight of the associated binary matrix E defined as follows.

Definition 5: [Crisscross weight of E] The crisscross weight of an erasure pattern E is the minimum size $|(X, Y)|$ over all possible covers (X, Y) of the associated binary matrix E. We denote the crisscross weight of E as $\mathsf{wt}_c(E)$. Note that a minimum-size cover of a given matrix E is not always unique. Further, the minimum size of a cover of a binary matrix is equal to the maximum number of 1's that

can be chosen in that matrix such that no two are on the same row or column [39, Theorem 5.1.4].

We characterize mixed erasure patterns corresponding to column and row failures of C that $\mathcal{C}$ can correct locally or globally. Recall that, for simplicity, we assume that the local codes associated with columns are disjoint in their support. We note that the proposed construction indeed results in disjoint local codes.

Proposition 1: Let $\mathcal{C}$ be an $(m \times n, k, d)$ rank-metric code with (r, δ) rank-locality. Let $\mathcal{C}_j$, $1 \leq j \leq \mu$, be the local $(r + \delta - 1, r, \delta)$ rank-metric code, and let C_j be the corresponding local array. Note that the size of each local array is $m \times r + \delta - 1$. Let E be the binary matrix associated with an erasure pattern on C with crisscross weight $\mathsf{wt}_c(E)$. If $\mathsf{wt}_c(E(C_j)) \leq \delta - 1$, then $E(C_j)$ can be corrected locally in C_j by accessing the unerased symbols only from C_j. Further, if $\mathsf{wt}_c(E) \leq d - 1$ and if $\mathsf{wt}_c(E(C_j)) \geq \delta$ for at least one local array, then E can be corrected globally by accessing all the unerased symbols of C.

Proof: The proof follows from the fact that a rank-metric code $\mathcal{C}$ of rank-distance d can correct any erasure pattern E such that $\mathsf{wt}_c(E) \leq d - 1$. To see this, suppose that $\mathsf{wt}_c(E) = t$. Consider a minimum-size cover (X, Y) of E. Delete the rows and columns indexed respectively by X and Y in all the codeword matrixes of $\mathcal{C}$. The resulting array code composed of matrices of size $m - |X| \times n - |Y|$ has rank-distance at least $d - t$. Thus, if $t \leq d - 1$, the deletion of rows and columns is injective and it is possible to recover the full array uniquely from the non-erased symbols. $\blacksquare$

Example 3: Suppose the data is to be stored on a 9×9 bit array C using the $(9 \times 9, 5, 5, 2, 2)$ rank-metric code discussed in Example 2. Note that the first three columns of C form the first local array C_1, the next three columns form the second local array C_2, and the remaining three columns form the third local array C_3. The encoding satisfies the correctability constraints mentioned in Example 1. We give a few examples of erasure patterns that are correcrable in Fig. 3, where locally correctable erasures are denoted as '?', while globally correctable erasures are denoted as '??'.

Remark 4: In Proposition 1, we only characterize the erasure patterns that are locally or globally correctable. It is interesting to consider efficient decoding algorithms on the lines of [40], [41].

Remark 5: We note that an $(m \times n, k, d, r, \delta)$ code may correct a number of erasure patterns that are not covered by the class mentioned in Proposition 1. This is analogous to the fact that an LRC can correct a large number of erasures beyond minimum distance. In fact, the class of LRCs that have the maximum erasure correction capability are known as maximally recoverable codes. Along similar lines, it is interesting to extend the notion of maximal recoverability for rank-metric and characterize all the erasure patterns that an $(m \times n, k, d, r, \delta)$ can correct.

REFERENCES

[1] A. Rowstron and P. Druschel, "Storage management and caching in past, a large-scale, persistent peer-to-peer storage utility," *SIGOPS Oper. Syst. Rev.*, vol. 35, no. 5, pp. 188–201, Oct. 2001.

[2] S. Ghemawat, H. Gobioff, and S.-T. Leung, "The google file system," *SIGOPS Oper. Syst. Rev.*, vol. 37, no. 5, pp. 29–43, Oct. 2003.

[3] C. Huang, H. Simitci, Y. Xu, A. Ogus, B. Calder, P. Gopalan, J. Li, and S. Yekhanin, "Erasure coding in windows azure storage," in *Proceedings of the 2012 USENIX Conference on Annual Technical Conference*, ser. USENIX ATC'12, 2012.

[4] S. Muralidhar, W. Lloyd, S. Roy, C. Hill, E. Lin, W. Liu, S. Pan, S. Shankar, V. Sivakumar, L. Tang, and S. Kumar, "F4: Facebook's warm BLOB storage system," in *Proceedings of the 11th USENIX Conference on Operating Systems Design and Implementation*, ser. OSDI'14, 2014, pp. 383–398.

[5] A. G. Dimakis, P. B. Godfrey, M. Wainwright, and K. Ramachandran, "Network Coding for Distributed Storage Systems," *Information Theory, IEEE Transactions on*, vol. 56, no. 9, pp. 4539–4551, Sep. 2010.

[6] A. G. Dimakis, K. Ramchandran, Y. Wu, and C. Suh, "A Survey on Network Codes for Distributed Storage," *Proceedings of the IEEE*, vol. 99, no. 3, pp. 476–489, Mar. 2011.

[7] K. V. Rashmi, N. B. Shah, and P. V. Kumar, "Optimal exact-regenerating codes for distributed storage at the msr and mbr points via a product-matrix construction," *IEEE Transactions on Information Theory*, vol. 57, no. 8, pp. 5227–5239, Aug 2011.

[8] C. Huang, M. Chen, and J. Li, "Pyramid codes: Flexible schemes to trade space for access efficiency in reliable data storage systems," in *Network Computing and Applications, 2007. NCA 2007. Sixth IEEE International Symposium on*, July 2007, pp. 79–86.

[9] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, "On the locality of codeword symbols," *Information Theory, IEEE Transactions on*, vol. 58, no. 11, pp. 6925–6934, Nov 2012.

[10] F. Oggier and A. Datta, "Self-repairing homomorphic codes for distributed storage systems," in *INFOCOM, 2011 Proceedings IEEE*, April 2011, pp. 1215–1223.

[11] M. Sathiamoorthy, M. Asteris, D. Papailiopoulos, A. G. Dimakis, R. Vadali, S. Chen, and D. Borthakur, "Xoring elephants: novel erasure codes for big data," in *Proceedings of the 39th international conference on Very Large Data Bases*, ser. PVLDB'13, 2013, pp. 325–336.

[12] I. Tamo and A. Barg, "A family of optimal locally recoverable codes," *Information Theory, IEEE Transactions on*, vol. 60, no. 8, pp. 4661–4676, Aug 2014.

[13] P. Delsarte, "Bilinear forms over a finite field, with applications to coding theory," *Journal of Combinatorial Theory, Series A*, vol. 25, no. 3, pp. 226 – 241, 1978.

[14] E. M. Gabidulin, "Theory of codes with maximum rank distance," *Problems Inform. Transmission*, vol. 21, no. 1, pp. 1–12, Jul 1985.

[15] R. M. Roth, "Maximum-rank array codes and their application to crisscross error correction," *IEEE Transactions on Information Theory*, vol. 37, no. 2, pp. 328–336, Mar 1991.

[16] M. Blaum and J. Bruck, "Mds array codes for correcting a single criss-cross error," *IEEE Transactions on Information Theory*, vol. 46, no. 3, pp. 1068–1077, May 2000.

[17] D. Ford, F. Labelle, F. Popovici, M. Stokely, V.-A. Truong, L. Barroso, C. Grimes, and S. Quinlan, "Availability in globally distributed storage systems," in *Proceedings of the 9th USENIX Symposium on Operating Systems Design and Implementation*, 2010.

[18] S. Nath, H. Yu, P. B. Gibbons, and S. Seshan, "Subtleties in tolerating correlated failures in wide-area storage systems," in *Proceedings of the 3rd Conference on Networked Systems Design & Implementation - Volume 3*, ser. NSDI'06, 2006, pp. 17–17.

[19] J. L. Hafner, V. Deenadhayalan, W. Belluomini, and K. Rao, "Undetected disk errors in raid arrays," *IBM J. Res. Dev.*, vol. 52, no. 4, pp. 413–425, Jul. 2008.

[20] M. Balakrishnan, A. Kadav, V. Prabhakaran, and D. Malkhi, "Differential raid: Rethinking raid for ssd reliability," *Trans. Storage*, vol. 6, no. 2, pp. 4:1–4:22, Jul. 2010.

[21] M. Blaum, J. L. Hafner, and S. Hetzler, "Partial-mds codes and their application to raid type of architectures," *IEEE Transactions on Information Theory*, vol. 59, no. 7, pp. 4510–4519, July 2013.

[22] K. W. Shum and Y. Hu, "Cooperative regenerating codes," *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7229–7258, Nov 2013.

[23] A. S. Rawat, A. Mazumdar, and S. Vishwanath, "On cooperative local repair in distributed storage," in *Information Sciences and Systems (CISS), 2014 48th Annual Conference on*, March 2014, pp. 1–5.

[24] P. Gopalan, C. Huang, B. Jenkins, and S. Yekhanin, "Explicit maximally recoverable codes with locality," *IEEE Transactions on Information Theory*, vol. 60, no. 9, pp. 5245–5256, Sept 2014.

[25] N. Prakash, V. Lalitha, and P. Kumar, "Codes with locality for two erasures," in *Information Theory (ISIT), 2014 IEEE International Symposium on*, June 2014, pp. 1962–1966.

[26] P. Gopalan, G. Hu, S. Saraf, C. Wang, and S. Yekhanin, "Maximally recoverable codes for grid-like topologies," *CoRR*, vol. abs/1605.05412, 2016. [Online]. Available: http://arxiv.org/abs/1605.05412

[27] J. S. Plank and M. Blaum, "Sector-disk (sd) erasure codes for mixed failure modes in raid systems," *Trans. Storage*, vol. 10, no. 1, pp. 4:1–4:17, Jan. 2014.

[28] M. Blaum, J. S. Plank, M. Schwartz, and E. Yaakobi, "Construction of partial mds and sector-disk codes with two global parity symbols," *IEEE Transactions on Information Theory*, vol. 62, no. 5, pp. 2673–2681, May 2016.

[29] J. Colgrove, J. D. Davis, J. Hayes, E. L. Miller, C. Sandvig, R. Sears, A. Tamches, N. Vachharajani, and F. Wang, "Purity: Building fast, highly-available enterprise flash storage from commodity components," in *Proceedings of the 2015 ACM SIGMOD International Conference on Management of Data*, ser. SIGMOD '15, 2015, pp. 1683–1694.

[30] S.-W. Jun, M. Liu, S. Lee, J. Hicks, J. Ankcorn, M. King, S. Xu, and Arvind, "Bluedbm: An appliance for big data analytics," *SIGARCH Comput. Archit. News*, vol. 43, no. 3, pp. 1–13, Jun. 2015.

[31] J. Han and L. Lastras-Montao, "Reliable memories with subline accesses," in *Information Theory, 2007. ISIT 2007. IEEE International Symposium on*, June 2007, pp. 2531–2535.

[32] N. Silberstein, A. Rawat, O. Koyluoglu, and S. Vishwanath, "Optimal locally repairable codes via rank-metric codes," in *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, July 2013, pp. 1819–1823.

[33] S. Goparaju and R. Calderbank, "Binary cyclic codes that are locally repairable," in *Information Theory (ISIT), 2014 IEEE International Symposium on*, June 2014, pp. 676–680.

[34] P. Huang, E. Yaakobi, H. Uchikawa, and P. H. Siegel, "Binary linear locally repairable codes," *CoRR*, vol. abs/1511.06960, 2015. [Online]. Available: http://arxiv.org/abs/1511.06960

[35] F. J. MacWilliams and N. J. A. N. J. A. Sloane, *The theory of error correcting codes*, ser. North-Holland mathematical library. Amsterdam, New York: North-Holland Pub. Co. New York, 1977.

[36] R. Lidl and H. Niederreiter, *Finite fields*, ser. Encyclopedia of mathematics and its applications. New York: Cambridge University Press, 1997.

[37] N. Prakash, G. Kamath, V. Lalitha, and P. Kumar, "Optimal linear codes with a local-error-correction property," in *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*, July 2012, pp. 2776–2780.

[38] A. Rawat, O. Koyluoglu, N. Silberstein, and S. Vishwanath, "Optimal locally repairable and secure codes for distributed storage systems," *Information Theory, IEEE Transactions on*, vol. 60, no. 1, pp. 212–236, Jan 2014.

[39] M. Hall, Jr., *Combinatorial Theory (2Nd Ed.)*. New York, NY, USA: John Wiley & Sons, Inc., 1998.

[40] E. M. Gabidulin and N. I. Pilipchuk, "Error and erasure correcting algorithms for rank codes," *Designs, Codes and Cryptography*, vol. 49, no. 1, pp. 105–122, 2008.

[41] D. Silva and F. R. Kschischang, "Fast encoding and decoding of gabidulin codes," in *2009 IEEE International Symposium on Information Theory*, June 2009, pp. 2858–2862.